Random Processes

Date: 27/02/2014

PhD Qualifying Examination

Full marks - 100

I. INSTRUCTIONS

1) Write your name and the question number on top of EVERY page.
2) Each question carry 20 marks. The sum on within the square braces represents the marks distribution in each question.
3) **Non-trivial step jumps will cause marks deduction; very big step jumps may lead to no credits for the problem**.
4) Only a yes/no answer to any question without a proof will not be considered for partial marks.

Q.1 Prove De Morgan's law, i.e., for any two arbitrary sets A and B, [10]

$$(A \cup B)^c = A^c \cap B^c.$$

Q.2 Prove the following:

- $P[\phi] = 0$, where ϕ is the empty set.
- If A and B are independent, then so are A and B^c.
- $P[A \cap B^c] + P[B] \leq 1$. [3+3+4]

You may use any of the probability Axioms and/or Theorems, but you have to provide appropriate statements of the results you use.

Q.3 Prove that

$$P[A_1] + P[A_2] + P[A_3] \geq P[A_1 \cup A_2 \cup A_3],$$

where A_1, A_2 and A_3 are arbitrary sets which may or may not be mutually disjoint. [10]

Q.4 Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time. What is $P[-|H]$, the conditional probability that a person tests negative given that the person does have HIV virus? What is $P[H|+]$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive? [10]

Q.5 A random binary source generates one bit per second with the following probability: 1 with probability ρ and 0 with probability $(1 - \rho)$. The bits generated in different times are independent of each other. Consider the random experiment of observing a sequence of bits of length 6, i.e., a typical output in the sample space is "010111".

- What is the probability of observing the sequence "000011"?
- What is the probability of observing exactly two 1's?
- what is the probability of observing three or more 1's? [3+3+4]

Q.6 a) Let X be an exponential random variable, i.e., its PDF is given by the following expression:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise,} \end{cases}$$

where $\lambda > 0$.

- Prove that $E(X) = \frac{1}{\lambda}$.
- Let B be the event that X is larger than or equal to 10, i.e., $B = \{X \geq 10\}$. Given B, find the conditional PDF of X, i.e., find $f_{X|B}(x)$. [5+5]

Q.7 Let Y be a random variable with the following PDF: [5+2+3]

$$f_Y(y) = \begin{cases} cye^{-\frac{y}{3}} & y \geq 0; \\ 0 & \text{otherwise} \end{cases}$$

- Find the value of c so that $f_Y(y)$ is a valid PDF.
- Compute $P[3 < Y \leq 5]$.
- Compute $P[Y > 4]$.

Q.8 Let X be a random variable with the following CDF: [2+3+5]

$$F_X(x) = \begin{cases} 0, & x < -1; \\ \frac{x}{4} + \frac{1}{2}, & -1 \leq x < 2; \\ 1, & x \geq 2, \end{cases}$$

- Is it a discrete/continuous/mixed random variable? Why?
- Write a single expression in terms of unit step functions for $F_X(X)$.
- Compute its PDF $f_X(x)$.

Q.9 Let X be a random variable with CDF and PDF denoted as $F_X(x)$ and $f_X(x)$, respectively. [7+3]

- Find the CDF and PDF of the derived random variable $Y = aX + b$, where both $b, a > 0$ in terms of CDF and PDF of X.
- Find the expected value of Y in terms of $E[X]$.

Q.10 The time between telephone calls at a telephone switch is an exponential random varable T with expected value 0.01. Given $T > .02$,

- What is $E[T|T > .02]$, the conditional expected value of T? [10]