Math 720 – Preliminary Exam

January 2025

Time: 120 mins.

1. Write your student ID number at the top of each page of your exam solutions.
2. Write only on the front page of each solution sheet.
3. Start each question on a new sheet of paper. Each question is worth 10 points.
4. In answering any part of a question, you may assume the results in the previous parts.
5. To receive full credit, answers must be justified.
6. You can do the problems in any order! If you get stuck, move on and come back to it.
7. In this exam, "ring" means "ring with unit" and "module" means "unital (unitary) module". Further, if $\phi : R \to S$ is a ring homomorphism, we assume $\phi(1_R) = 1_S$.

Student ID Number: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. (10 points) Let G be a $\mathbb{Z}$-module with generators x, y, z subject to the relations

$$x + 2z = 0, \; x - 2y - 4z = 0, \text{ and } 3x - 4y = 0.$$

Find the elementary divisors of the $\mathbb{Z}$-module G.

2. (10 points) As a quotient of $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}/6\mathbb{Z}$-module. Show that $\mathbb{Z}/2\mathbb{Z}$ is projective but not free as a $\mathbb{Z}/6\mathbb{Z}$-module.

3. (10 points) Prove that $\frac{\mathbb{Q}[x]}{(x)} \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x+1)} \cong (0)$.

4. (10 points) Let k be a field, and let $f(x) \in k[x]$ be a polynomial of degree n.
   (a) Prove that if $f(x)$'s irreducible factorization has a repeated factor, then there are at least two similarity classes of matrices whose characteristic polynomial is $f(x)$.
   (b) Prove that if $f(x)$'s irreducible factorization does NOT has a repeated factor, then there is a unique similarity class of matrices whose characteristic polynomial is $f(x)$.

5. (10 points) Let p be a prime number and consider the following set

$$A = \left\{ \frac{a}{b} \in \mathbb{Q} \; : \; p \nmid b \right\}.$$

A is a subring of $\mathbb{Q}$ (you do not need to prove this).
   (a) Find all multiplicative units of the subring A.
   (b) Use part (a) to find all prime ideals of A.

6. (10 points) **Definition:** An ideal I of a commutative ring R is said to be *semiprime* if for all ideals J of R, $J^2 \subseteq I$ implies $J \subseteq I$.

   Definition: Similarly, an ideal I of a commutative ring R is said to be *radical* if for all $r \in R$, $r^k \in I$ for some positive integer k implies $r \in I$.

   Prove that the zero ideal is radical if and only if it is semiprime.

7. (10 points) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.