Preliminary Examination (Math 726)
January 2025

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam, "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\phi : R \to S$ is a ring homomorphism, we also assume $\phi(1_R) = 1_S$.

1. Consider $M = \mathbb{Z}/4\mathbb{Z}$ and $N = \mathbb{Z}/6\mathbb{Z}$ as $R = \mathbb{Z}/24\mathbb{Z}$ modules.

(a) Write a projective resolution of M as an R-module.

(b) Compute $\mathrm{Tor}_i^R(M, N)$ for all $i \geq 0$.

(c) Compute $\mathrm{Ext}^i_R(M, N)$ for all $i \geq 0$.

2. We say an object F in a category C is *final* if $\mathrm{Hom}_{\mathsf{C}}(A, F)$ is a singleton for every $A \in \mathrm{ob}(\mathsf{C})$. Show that final objects are unique up to isomorphism.

3. Let R be a commutative ring. Prove that if x, y is a regular sequence in R and y is regular, then y, x is also a regular sequence.

4. Let R be a commutative ring and suppose there is a short exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow N \longrightarrow 0$$

for some $n \in \mathbb{N}$. Suppose further that $\mathrm{Ext}^1_R(N, M) = 0$. Prove that M and N are projective.

5. Let $(R, \mathfrak{m}, k)$ be commutative, Noetherian, local ring, and suppose $\mathrm{Tor}_d^R(k, k) = 0$ for some $d \in \mathbb{N}$. Prove that for every finitely generated R-module M, $\mathrm{pdim}_R(M) < d$.