Algebra Preliminary Examination

May 2019

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- For this exam you have two options:
  - (i) Submit solutions to questions from part A and from part B.
  - (ii) Submit solutions to questions from part A and from part C.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1_R) = 1_S$.
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let R be the polynomial ring $\mathbb{Q}[x, y]$ in two indeterminates x, y over the field $\mathbb{Q}$ of rational numbers. Let I be the 2-generated ideal (x, y) in the ring R. **Prove or disprove**:
   - (a) I is a maximal ideal of the ring R.
   - (b) I can be principally generated as an ideal of the ring R.
2. Let R be the ring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ of Gaussian integers. Prove that if I is any nonzero ideal of R, then R/I is a finite ring.
3. Let R be a ring and let M be an R-module. For submodules $L, K \leq M$ let $K \oplus L$ denote their *external* direct sum. Construct a short exact sequence
$$0 \to K \cap L \to K \oplus L \to K + L \to 0.$$
4. Let $R \subseteq S$ be an extension of rings and let P be a projective R-module. Prove that $S \otimes_R P$ is a projective S-module.
5. Let $\mathbb{F}$ be a field and let V, W be finite dimensional $\mathbb{F}$-vector spaces. Fix any subspace $U \leq V$ and prove that the following statements are equivalent.
   - (i) There exists a linear transformation $T : V \to W$ such that $U = \ker T$.
   - (ii) $\dim(V) \leq \dim(U) + \dim(W)$.
6. Let $T : \mathbb{Q}^7 \to \mathbb{Q}^7$ be a linear transformation with minimal polynomial $m_T(x) = (x^2 + 2)(x + 2)^3$. Find all possible rational canonical forms for T.

B. Groups, Fields, and Galois Theory (option 1)

1. Recall that S_n is the permutation group on the set $\{1, 2, ..., n\}$ and A_n is the subgroup consisting of all the even permutations. Prove that if $n \geq 5$ then A_n is the only proper nontrivial normal subgroup of S_n.

2. Let G be an infinite group with a nonidentity element $a \in G$ such that the conjugacy class $O(a) = \{gag^{-1} : g \in G\}$ is finite. Prove that G is not a simple group.

3. Let G be a group of order $160 = 2^5 5$. Prove that if G has two distinct groups of order 80, then G has a normal Sylow 5-subgroup.

4. Let $\omega \in \mathbb{C}$ be a primitive 7^{th} root of unity and let $K = \mathbb{Q}(\omega)$.

   (a) Determine the Galois group $\text{Gal}(K/\mathbb{Q})$.

   (b) How many intermediate fields lie between $\mathbb{Q}$ and K? Justify your answer.

C. Homological Algebra (option 2)

1. Let R be an integral domain and Q its field of fractions. Let M be an R-module. Prove that $\text{Tor}_1^R(Q/R, M) \cong T(M)$, the torsion submodule of M.

2. Let R be a noetherian ring, M a finitely generated R-module. Prove that M is a projective R-module if and only if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R.

3. If P is a finitely generated projective module over a ring R, show that $\text{Hom}_R(P, R)$ is a projective R-module.

4. Let $R = k[X, Y]/(XY)$ where k is a field and $M = R/XR$. Prove that $\text{pd}_R\, M = \infty$.