Preliminary Examination (Math 720)
August 2022

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\varphi : R \to S$ is a ring homomorphism, we also assume $\varphi(1_R) = 1_S$

1. Let R be an integral domain. Prove that if every descending chain

$$(r_1) \supseteq (r_2) \supseteq (r_3) \supseteq \dots$$

of principal ideals in R stabilizes, then R is a field.

2. Let R be an integral domain and let $\pi \in R$ be an irreducible element that is not prime.

(a) Prove that there exists an element $r \in R$ such that the 2-generated ideal (π, r) is not a principal ideal. **Hint**: There exist elements $a, b \in R$ such that $ab \mid \pi$ but $\pi \nmid a$ and $\pi \nmid b$.

(b) Construct a 2-generated ideal in the ring $\mathbb{Z}[\sqrt{-5}]$ that is not principal.

3. Suppose that $g, h \in \mathbb{Q}[X]$. Use Gauss's Lemma to prove that if $f = gh$ belongs to $\mathbb{Z}[X]$, then the product of any coefficient of g with any coefficient of h belongs to $\mathbb{Z}$.

4. Let A, B be finite rings with at least two elements and let $R = A \times B$. Hence, A is an R-module via the natural ring homomorphism $R \to A$ projecting onto the first coordinate. Prove or disprove each statement:

(a) A is a projective R-module.

(b) A is a free R-module.

5. Let R be an integral domain with field of fractions K and let I be an ideal of R. Prove that $(R/I) \otimes_R K = 0$ if and only if $I \neq 0$.

6. Let F be a field and let U, V, W be finite dimensional F-vector spaces with linear transformations $S : U \to V$ and $T : V \to W$. Prove that

$$\dim_F(\ker(T \circ S)) \leq \dim_F(\ker T) + \dim_F(\ker S)$$

7. Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements and let V be an $\mathbb{F}_2$-vector space such that $\dim(V) = 3$. Find all possible rational canonical forms for a linear transformation $T : V \to V$ satisfying $T^6 = 1$.