Preliminary Examination (Math 721)
August 2022

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.

1. Recall that D_n is the dihedral group acting on n vertices. Prove that D_n has a subgroup of order 4 if and only if n is even.

2. We call a subgroup $N \subseteq G$ a *maximal normal subgroup* provided (i) N is a proper normal subgroup of G, and (ii) if H is another normal subgroup G such that $N \subseteq H \subseteq G$, then either $N = H$ or $H = G$.

   Prove the following statements:

   (a) N is a maximal normal subgroup of G if and only if G/N is a simple group.

   (b) If M and N are distinct maximal normal subgroups of G, then $M \cap N$ is a maximal normal subgroup of both M and N.

3. Classify all groups of order 45.

4. Let N be a normal subgroup of the group G.

   (a) Prove that the map $G \times N \to N$ given by $g \cdot n = gng^{-1}$ is a well-defined group action.

   (b) Suppose now that G is a finite p-group and that N is a nontrivial normal subgroup. Prove that $N \cap Z(G)$ is a nontrivial subgroup of G. Here, $Z(G)$ denotes the center of G.

5. Let $F \subseteq K$ be an extension of fields. **Prove or disprove** each statement:

   (a) If $[K : F]$ is prime, then $K = F(u)$ for *every* element $u \in K - F$.

   (b) If $[K : F]$ is prime, $F \subseteq K$ is a Galois extension.

6. Recall that a field F is called *perfect* if every irreducible polynomial $p(x) \in F[x]$ is separable (i.e., has no repeated roots). Prove that if F is a perfect field and the extension $F \subseteq K$ of fields is algebraic, then K is perfect.

7. Let $f(x) \in \mathbb{Q}[x]$ be a cubic polynomial with splitting field K (over $\mathbb{Q}$). Prove that if $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}_3$, then all the roots of f must belong to $\mathbb{R}$.