Preliminary Examination (Math 726)

August 2022

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\varphi : R \to S$ is a ring homomorphism, we also assume $\varphi(1_R) = 1_S$

1. Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider $M = \mathbb{Z}/2\mathbb{Z}$ as an R-module.

(a) Write a projective resolution of M.

(b) Compute $\mathrm{Tor}^i_R(M, M)$ for all $i \geq 0$.

(c) Compute $\mathrm{Ext}^i_R(M, M)$ for all $i \geq 0$.

2. Let R be a commutative noetherian ring and M a finitely generated R- module. Prove that there exists a projective resolution of M in which all modules are finitely generated.

3. Let R be a noetherian commutative ring and M a finitely generated R-module. Let a_1, a_2 be an M-regular sequence such that $I = (a_1, a_2) \subseteq \mathrm{Jac}(R)$. Prove that a_2 is an M-regular element.

4. Let R be a commutative ring, M an R-module, and x an M-regular element. Assume that we have an exact sequence of R-modules

$$N_2 \to N_1 \to N_0 \to M \to 0.$$

Prove that the induced sequence

$$N_2/xN_2 \to N_1/xN_1 \to N_0/xN_0$$

is also exact.

5. Let R be a commutative ring, M an R-module and $x \in R$ an element that is both R-regular and M-regular. Prove that

$$\mathrm{pd}_{R/xR}\, M/xM \leq \mathrm{pd}_R\, M.$$