Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation. Answers may include binomial and multinomial coefficients, but should not be written in terms of Stirling or Bell numbers.

Shorter questions: (5 points each)

1. How many ways can the letters in “BLABBING” be arranged so that none of the B’s are adjacent?

2. Find the closed form ordinary generating function for the sequence,

$$a_n = \begin{cases} 2^n, & n \text{ even}, n \geq 0, \\ 2^n - 1, & n \text{ odd}, n \geq 1. \end{cases}$$

3. Joe has a list of k distinct tasks to distribute to his kids Boe, Doe, Loe, Noe, and Trish. How many ways can he distribute the tasks so that Trish gets assigned at least 2 tasks?

4. I have 10 indistinguishable balls, and I will paint each one either red, green, yellow, or blue. The order in which the balls are painted does not matter. In how many ways can I paint the balls so that at least one is painted red?

5. Your combinatorics class has n distinct students, and the final project requires students to work in 5 groups of size 1 or more. How many ways can the groups be formed?

Longer questions: (10 points each)

6. How many lattice paths start at $(0, 0)$, pass through $(4, 6)$, end at $(10, 10)$, and never go below the line $y = x$?

7. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with n parts, and let $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ be the conjugate partition to λ. Prove the following identity by filling Young diagrams with appropriate non-negative integers:

$$\sum_i \binom{\lambda_i}{2} = \sum_i (i-1)\lambda'_i$$

8. Let d_n be the number of derangements of length n. Give a combinatorial proof of the following:

$$d_{n+1} = n(d_n + d_{n-1})$$

9. Prove the following identity:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{k=0}^{\infty} x^{k^2} \left[\prod_{\ell=1}^{k} \frac{1}{1-x^\ell} \right]^2$$

10. How many injective functions $f : [n] \to [3n]$ satisfy $f(i) \notin \{i, i+n, i+2n\}$ for $i = 1, \ldots, n$?