ALGEBRA PRELIMINARY EXAMINATION

SPRING 2002

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word "domain" means "integral domain," and "PID" means "principal ideal domain."

(1) Show that there is no simple group of order 96.

(2) Show that if α is a root of the polynomial $f(x) = x^5 - x^3 + 1$ and $\mathbb{F} = \mathbb{Q}(\alpha)$ then $\mathbb{F}$ is not Galois over $\mathbb{Q}$.

(3) Let R be a commutative ring with identity and P a projective $R-$module and N a free $R-$module. Show that $P \otimes_R N$ is also a projective $R-$module.

(4) Show any finite subgroup of the multiplicative group of a field is cyclic.

(5) Suppose that R is a non-Noetherian ring. Show that there is a prime ideal $\wp \subseteq R$ such that $\wp$ is not finitely generated.

(6) Suppose that G is a group and x is an element with precisely two distinct conjugates (the number of distinct elements in the set $\{g^{-1}xg | g \in G\}$ is 2). Show that G possesses a nontrivial normal subgroup.

(7) Suppose that G is a finite group and for each prime p dividing the order of G the Sylow $p-$subgroup is normal. Show that if the order of G is not divisible by any cube (other than 1), then G is abelian.

(8) Let T be a torsion abelian group (that is, every element is of finite order) and let $\mathbb{Q}$ denote the additive group of rational numbers. Show that $T \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

(9) Consider the following commutative diagram of $R-$modules you may assume that R is commutative with identity and all modules are unitary) with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

Show that if f and h are one to one, then so is g.

(10) Find all monic irreducible polynomials of degree 2 over the field $\mathbb{F}_3$ of three elements. For each irreducible $p(x)$ that you find, determine the ring structure of the quotient ring $\mathbb{F}_3[x]/(p(x))$. How many isomorphism classes are there?