ALGEBRA PRELIMINARY EXAMINATION

SPRING 2002

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word "domain" means "integral domain," and "PID" means "principal ideal domain." S_n denotes the symmetric group on n objects.

(1) Show there is no subgroup of S_5 of order 40.

(2) Let $f(x)$ be a irreducible cubic polynomial over the rational numbers $\mathbb{Q}$ such that $f'(x)$ has no real roots and let $\mathbb{F}$ be a splitting field of $f(x)$ over $\mathbb{Q}$. Compute $[\mathbb{F} : \mathbb{Q}]$.

(3) Show that a unitary $R-$module, I, is injective if and only if I is a direct summand of any module of which it is a submodule (that is, if $I \subseteq M$ then there is a submodule $J \subseteq M$ such that $M = I \oplus J$). You may assume for one direction that any module may be embedded in an injective $R-$module.

(4) Let K be an algebraic field extension of the field F and let D be an integral domain such that $F \subseteq D \subseteq K$. Show that D is a field.

(5) Show that any group of order p^2q (with p and q distinct primes) is solvable.

(6) Suppose that G is a group of order 96. Show that G necessarily has a normal subgroup of order 32 or 16 (hint: a group acts transitively on its Sylow subgroups).

(7) Let R be a commutative ring with identity and let $\mathfrak{M} \subseteq R$ be a maximal ideal. Show that any ideal of the form $(\mathfrak{M}, x)$ is maximal in the polynomial ring $R[x]$. Show that the converse is *never* true (i.e. show that in $R[x]$ there are always maximal ideals that are not of the form $(\mathfrak{M}, x)$).

(8) Let R be a UFD with only finitely many nonzero primes (up to units) $\{p_1, p_2 \cdots, p_t\}$. Show that R is a PID. Also show that, if addition, $t = 1$ then R is a Euclidean domain. (You may use the fact that R is a PID if and only if every prime ideal of R is principal).

(9) Suppose that $f : M \longrightarrow N$ is an injective homomorphism of $R-$modules and $1_D : D \longrightarrow D$ is the identity map on another $R-$module, D. Give an example to show that the map $f \otimes 1_D : M \otimes_R D \longrightarrow N \otimes_R D$ need not be injective.

(10) Show that every (unitary) $R-$module is the homomorphic image of a free $R-$module.