ALGEBRA PRELIMINARY EXAMINATION

AUGUST 2003

Notes. $\mathbb{Z}$ and $\mathbb{Q}$ refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word "domain" means "integral domain," and "PID" means "principal ideal domain."

(1) Classify all groups of order p^2 and show that your classification is complete.

(2) Show that every group of order 56 has either a normal subgroup of order 8 or a normal subgroup of order 7.

(3) If G is a group show that the subgroup generated by

$$xyx^{-1}y^{-1} \qquad x, y \in G$$

is a normal subgroup of G.

(4) Let R be a commutative domain. Show that every $R-$module is projective if and only if every $R-$module is injective.

(5) Let R be a commutative ring with identity P and Q be projective $R-$modules. Show that $P \otimes_R Q$ is a projective $R-$module.

(6) Let R be a commutative domain. We say that R has the "cyclic submodule property" if all submodules of any cyclic $R-$module are cyclic. Show that R has the "cyclic submodule property" if and only if R is a PID.

(7) Let α and β be rational numbers. Show that $\mathbb{Q}(\sqrt{\alpha}) \cong \mathbb{Q}(\sqrt{\beta})$ (as fields) if and only if $\alpha\beta$ is a square in $\mathbb{Q}$.

(8) Let n be a prime and K the splitting field of $x^n - 1$ over $\mathbb{Q}$. Show that for all $a \in K$ either $x^n - a$ is irreducible in $K[x]$ or it factors into linear factors over $K[x]$.

(9) Let I be an invertible ideal of a quasi-local domain (that is, a domain with a unique maximal ideal). Show that I is principal, and then use this to show that if D is Dedekind and $\mathfrak{P}$ is a nonzero prime ideal, then $D_{\mathfrak{P}}$ is a Noetherian valuation domain (that is, a PID with a unique non-zero prime ideal).

(10) Let $\mathbb{F}$ be a field. Find all possible Jordan canonical forms for a square-nilpotent ($\mathbb{M}^2 = 0$) 4×4 matrix.