ALGEBRA PRELIMINARY EXAMINATION

MAY 2004

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and the rational numbers respectively. All rings are commutative with identity unless specifically indicated otherwise.

(1) Let G be a simple group and A a nonidentity abelian group. If $\phi : G \longrightarrow A$ is a surjection, then show that G is cyclic of prime order.

(2) Let G be a group of order pm where p is a prime number $p > m$. Show that G cannot be simple.

(3) Show that any finite group generated by two elements of order 2 is dihedral (here we will consider the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be dihedral).

(4) Show that an arbitrary intersection of prime ideals is a radical ideal.

(5) Recall that a ring is called Artinian if it satisfies the descending chain condition on ideals (any chain of ideals $I_1 \supseteq I_2 \supseteq I_3 \cdots$ stabilizes). Show that any Artinian domain is a field.

(6) Let G be a finite group and K a field such that $(\mathrm{char}(K), |G|) = 1$. The group algebra $K[G]$ is defined to be the set of formal sums $\sum_{i=1}^{m} k_i g_i$ with $k_i \in K$ and $g_i \in G$ (with $\sum_{i=1}^{m} k_i g_i + \sum_{i=1}^{m} t_i g_i = \sum_{i=1}^{m} (k_i + t_i) g_i$ and $(k_1 g_1)(k_2 g_2) = k_1 k_2 (g_1 g_2)$ and extend by using distributivity). Show that in $K[G]$ the element

$$\frac{1}{n} \sum_{g \in G} g,$$

where $n = |G|$, is idempotent.

(7) Show that the polynomial $x^4 - 2x^2 + 2$ is irreducible over $\mathbb{Q}$ and compute its Galois group (over $\mathbb{Q}$). Hint: optionally, you could recall the transitive subgroups of S_4 to eliminate a number of possibilities.

(8) Suppose R is commutative with identity, $I \subseteq R$ is an ideal, and M is an $R-$module. Show that there is an $R-$module isomorphism

$$R/I \otimes_R M \cong M/IM.$$

(9) Show that the $R-$module P is projective if and only if given any surjective homomorphism $\phi : B \longrightarrow C$, the induced homomorphism $\overline{\phi} : \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C)$ is surjective.

(10) Give two examples, in characteristic 2, of fields as follows. First give an example of an extension of fields that is not seperable and then give an example of a quadratic extension of fields that is separable.