Algebra Preliminary Examination

February 2009

Instructions: Begin each question on a new sheet of paper. Justify each of your answers. In this exam, each ring has identity and each module is unital.

1. Up to isomorphism, describe all abelian groups of order 2000.
2. Let G be a group. For each $g \in G$ define the map $\phi_g \colon G \to G$ by the formula $\phi_g(h) = g^{-1}hg$. Let $\operatorname{Aut}(G)$ denote the set of all automorphisms of G, which is a group under composition. Prove the following:
   (a) For each $g \in G$, the map ϕ_g is an automorphism of G. Such a map is called an *inner automorphism* of G.
   (b) The map $\Phi \colon G \to \operatorname{Aut}(G)$ given by $\Phi(g) = \phi_g$ is a homomorphism.
3. Give an example of a field extension that is not separable. Give an example of a field extension that is not normal.
4. Let $K \subseteq L$ be a finite field extension, and consider a polynomial $f \in K[X]$ such that $\deg(f)$ and $[L \colon K]$ are relatively prime. Prove that, if f is irreducible over K, then f is irreducible over L. Show that the hypothesis "$\deg(f)$ and $[L \colon K]$ are relatively prime" is necessary.
5. Consider the polynomial $f = X^4 + 10X + 5 \in \mathbb{Z}[X]$. Prove that the quotient $K = \mathbb{Q}[X]/(f)$ is a field extension of $\mathbb{Q}$ and compute $[K \colon \mathbb{Q}]$. Is the ring $\mathbb{Z}[X]/(f)$ a field?
6. Let R be a ring, and set $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$. Prove that $Z(R)$ is a subring of R. Must $Z(R)$ be a left ideal of R?
7. Let R be a commutative ring and let M be a noetherian R-module. Set $\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\}$ and prove that $R/\operatorname{Ann}_R(M)$ is a noetherian ring.
8. Let R be a commutative ring, let $I \subseteq R$ be an ideal, and let M be an R-module. For each element $m \in M$, set $\operatorname{Ann}_R(m) = \{r \in R \mid rm = 0\}$. Prove the following:
   (a) For each element $m \in M$, the set $\operatorname{Ann}_R(m) \subseteq R$ is an ideal.
   (b) There is an element $m \in M$ such that $I = \operatorname{Ann}_R(m)$ if and only if there is an R-module monomorphism $R/I \to M$.
9. Let R be a commutative ring and let M be an R-module. Prove that M is a finitely generated projective R-module if and only if there is an R-module Q and an integer $n \geqslant 0$ such that $P \oplus Q \cong R^n$.
10. Let R be a ring and consider the following diagram of left R-module homomorphisms wherein τ and π are the canonical epimorphisms:

$$\begin{array}{ccccc} M & \xrightarrow{f} & M' & \xrightarrow{\tau} & M'/\operatorname{Im}(f) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g''} \\ N & \xrightarrow{h} & N' & \xrightarrow{\pi} & N'/\operatorname{Im}(h). \end{array}$$

Assume that the left-hand square commutes, and prove that there is a left R-module homomorphism $g'' \colon M'/\operatorname{Im}(f) \to N'/\operatorname{Im}(h)$ making the right-hand square commute.