Algebra Preliminary Examination
June 2008

Instructions: Begin each question on a new sheet of paper.
In this exam, all rings have identity and all modules are unital.

1. Prove that there are no simple groups of order 520.
2. Let G be a finite abelian group, and let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Prove that $\mathrm{Aut}(G)$ is abelian if and only if G is cyclic.
3. Let S be an integral domain, and let R be a subring of S such that S is finitely generated as an R-module. Assume that R is a field, and prove that S is a field.
4. Prove that every principal ideal domain is a unique factorization domain. Give an example of a unique factorization domain that is not a principal ideal domain. Justify your response.
5. Find a polynomial $f \in \mathbb{Q}[x]$ such that $\deg(f) = 10 = [K : \mathbb{Q}]$ where K is a splitting field of f over $\mathbb{Q}$.
6. Let $K \subseteq L$ be a field extension of degree 2.
   (a) Prove that the extension $K \subseteq L$ is normal.
   (b) Find a field extension $K \subseteq L$ of degree 2 that is not separable. Justify your response.
7. Let K be a field of characteristic 0 and let L be a splitting field over K of the polynomial $x^n - a \in K[x]$. Prove that the Galois group $\mathrm{Gal}(L : K)$ is solvable.
8. Let R be a ring. Recall that an R-module $N \neq 0$ is *simple* if the only submodules of N are 0 and N. Let M be an R-module that is a sum of simple R-submodules. Prove that M is a direct sum of simple R-modules.
9. Classify all finitely generated $\mathbb{Z}/(6)$-modules up to isomorphism. Justify your response.
10. Let R be a principal ideal domain and let $a, b \in R$ be nonzero nonunits. Prove that $R/(a) \otimes_R R/(b) \cong R/(\gcd(a, b))$.