PH.D. PRELIMINARY EXAMINATION IN ALGEBRA

AUGUST 1995

Notation. In the following examination R denotes a ring. The integers are denoted by $\mathbb{Z}$, and the rational numbers are denoted by $\mathbb{Q}$.

Problems.

(1) Show that every group of order 15 is Abelian.

(2) Let G be a finite non-cyclic Abelian group. Show that for some p, the p-Sylow subgroup of G is non-cyclic.

(3) Let G be a group, let K be a normal subgroup of G and let H be a subgroup of K. Denote by $N_G(H) = \{a \in G \,|\, aHa^{-1} = H\}$, the normalizer of H in G. If for every $g \in G$, there exists $k \in K$ with $gHg^{-1} = kHk^{-1}$, then $G = KN_G(H)$.

(4) Show that no group of order 96 is simple.

(5) Let R be a commutative Noetherian ring. Show that $R[x]$ is a Noetherian ring.

(6) Let $\phi : M \to N$ be an epimorphism of R-modules, let P be a projective R-module and let

$$\overline{\phi} : Hom_R(P, M) \to Hom_R(P, N)$$

be the map given by $\overline{\phi} : g \mapsto \phi \circ g$. Show that $\overline{\phi}$ is an epimorphism of Abelian groups.

(7) Let R be a finite semisimple commutative ring. Show that R is a direct sum of fields.

(8) Suppose that $k/\mathbb{Q}$ is a Galois extension and K/k is a Galois extension, show that $K/\mathbb{Q}$ need not be a Galois extension.

(9) What is the Galois group of $x^3 - 2$ over $\mathbb{Q}$?

(10) Show that the ring of endomorphisms of a simple R-module is a division ring.