Algebra Preliminary Examination

February 2014

Directions: Show all work for full credit. Unless otherwise stated, R denotes a commutative ring with identity and all R-modules are unital. Good luck and just do the best you can.

1. Let S_n be the symmetric group of permutations on n letters and let A_n be the subgroup of even permutations. Prove that A_n is the only subgroup of S_n with index 2.

2. Let G be a group of order 105. Prove that G contains a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

3. Let I be a *proper* ideal of the ring R and let $S = 1 + I$. Prove that S is a multiplicatively closed subset of R and that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

4. Suppose that R is an integral domain. Prove that R is a PID if it satisfies the following two conditions:

   (i) Every pair of nonzero elements $r, s \in R$ has a greatest common divisor d which can be written in the form $d = rx + sy$ for some $x, y \in R$.

   (ii) If $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq ...$ is a chain of nonzero principal ideals, then there exists a positive integer N such that $(a_n) = (a_N)$ for all $n \geq N$.

5. Suppose that P and Q are projective R-modules. Prove that $P \otimes_R Q$ is a projective R-module.

6. Let M be a Noetherian R-module and let $\varphi : M \to M$ be an endomorphism of M. Prove that $\ker(\varphi^N) \cap \mathrm{Im}(\varphi^N) = 0$ for some positive integer N.

7. Suppose that V, W are finite dimensional vector spaces over the field F and let U be a subspace of V. Prove that there exists a linear transformation $\alpha \in \mathrm{Hom}(V, W)$ such that $\ker \alpha = U$ if and only if $\dim(U) \geq \dim(V) - \dim(W)$.

8. Let $\mathbb{F}_2$ denote the field with two elements and let V be an $\mathbb{F}_2$-vector space such that $\dim_{\mathbb{F}_2}(V) = 3$. How many possible rational canonical forms are there for a linear transformation $\theta \in \mathrm{Hom}(V, V)$ satisfying $\theta^6 = 1_V$? Justify your answer.

9. Find the minimum polynomial of the complex number $\sqrt{3 + 4i} + \sqrt{3 - 4i}$ over the field $\mathbb{Q}$ of rational numbers. Justify your answer.

10. Let F be a field of characteristic zero containing a primitive n^{th} root of unity and let $a \in F$. Prove that if K is the splitting field of the polynomial $p(x) = x^n - a$, then the Galois group $\mathrm{Gal}(K/F)$ is cyclic.