PH.D. PRELIMINARY EXAMINATION IN ALGEBRA

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Notation. In the following examination, G denotes a group, R denotes a ring, and the rational numbers are denoted by $\mathbb{Q}$.

Problems.

(1) Show that if G is a group with center $Z(G)$ then $G/Z(G)$ is cyclic if and only if $G/Z(G)$ is Abelian.

(2) Let p be the smallest prime dividing the order of a finite group. Show that every subgroup of index p is normal.

(3) Show that every group of order 35 is cyclic.

(4) Let G be a finite Abelian group. Let $\mathbb{QG}$ be the group algebra of G. Show that every irreducible $\mathbb{QG}$-module is isomorphic to a finite field extension of $\mathbb{Q}$.

(5) Let R be a commutative Noetherian ring. Show that $R[x]$ is a Noetherian ring.

(6) Show that E is an injective R-module if and only if for every monomorphism $\phi : M \to N$ of R-modules, the map:

$$\phi^* : Hom_R(N, E) \to Hom_R(M, E),$$

defined by $\phi^* : f \mapsto \phi \circ f$, is an epimorphism.

(7) Let K be a field. Show that there exists an algebraic closure of K.

(8) What is the Galois group of $x^5 - 7$ over $\mathbb{Q}$?

(9) Show that if ϕ is an endomorphism of a simple R-module then either ϕ is an isomorphism or else it is the zero map.

(10) Let G be a group and $x \in G$ be an element of order 2 and is the only element in G of order 2. Prove that there is a maximal subgroup of G which does not contain x.