Algebra Preliminary Examination

January 2010

Directions: Begin each question on a new sheet of paper. All rings are commutative with identity and all modules are unital.

1. Let R be a PID and let I be any proper nonzero ideal of R.

(a) Prove that $I = P_1P_2 \cdots P_n$ where $n \in \mathbb{Z}^+$ and each P_i is a prime ideal.

(b) Prove that if $I = Q_1Q_2 \cdots Q_m$, is any other prime ideal factorization of I, then $m = n$ and, after a suitable renumbering, $P_i = Q_i$ for each $i \leq n$.

2. Let G be a group with $N \lhd G$, $[G : N]$ finite, $H < G$, $|H|$ finite, and $\gcd([G : N], |H|) = 1$. Prove that $H < N$.

3. Suppose that R is a subring of a field K and that R contains a field F. Prove that if K/F is a finite field extension, then R is a field.

4. Let G be a group with $|G| = 231$. Show that $Z(G)$ contains a Sylow 11-subgroup of G and that G contains a normal Sylow 7-subgroup.

5. Let G be an infinite group. Prove that G is cyclic if and only if G is isomorphic to each of its proper subgroups.

6. Let D be an infinite integral domain. Prove that if D has a finite number of maximal ideals, then D must have an infinite number of units.

7. How many ideals are in the ring $\mathbb{Z}[x]/(2, x^3 + 1)$? Justify your answer.

8. Determine the Galois group of the following polynomials over the given field (the symbol ζ_n stands for a primitive nth root of unity).

a) $x^7 + 11$ over $\mathbb{Q}(\zeta_7)$,

b) $x^7 + 11$ over $\mathbb{R}$,

9. Suppose that A is a finite abelian group of order n and write $n = p^k m$ where p is a prime number and $\gcd(p, m) = 1$. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

10. Let R be a ring and let M be a finitely generated projective R-module. Prove that $\mathrm{Hom}(P, R)$ is a finitely generated projective R-module.