Algebra Preliminary Examination

June 2013

Directions: Show all work for full credit. Unless otherwise stated, R denotes a commutative ring with identity and M denotes a unital R-module. Good luck and just do the best you can.

1. Let G be a group of order 108. Show that G has a normal subgroup of order 27 or a normal subgroup of order 9.

2. Suppose that N is a normal subgroup of G. Prove that if G/N and N are both solvable groups, then G is a solvable group.

3. Find the invariant factor direct sum decomposition of a finitely generated abelian group G with generators $\{x, y, z\}$ subject to the relations

$$\begin{aligned} x + 2y + 5z &= 0 \\ 3x + 3y + 9z &= 0. \end{aligned}$$

4. Let R be a ring and let Σ be the set of all proper ideals of R that consist only of zero-divisors.
(a) Prove that Σ has maximal elements with respect to inclusion.
(b) Prove that every maximal element of Σ is a prime ideal.

5. Let s be an element of the ring R and let $S = \{s^n : n \geq 0\}$. Let $R[x]$ be the polynomial ring in one variable over R. Prove that there exits a ring isomorphism $S^{-1}R \simeq R[x]/(sx - 1)$.

6. Let M be a Noetherian R-module and let $\varphi : M \to M$ be an R-module homomorphism. Prove that if φ is surjective, then φ is bijective.

7. Let P be a finitely generated projective R-module. Prove that $\mathrm{Hom}(P, R)$ is a finitely generated projective R-module.

8. Let V be a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and let $\theta \in \mathrm{Hom}(V, V)$.
(a) Prove that if $\theta^3 = I$, then θ is diagonalizable.
(b) Does the result in (a) hold if the field $\mathbb{C}$ is replaced by $\mathbb{Q}$? Justify your answer.

9. Consider the polynomial $p(x) = x^3 + x + 1$ in $\mathbb{F}_2[x]$ and let α be a root of $p(x)$ in some extension of $\mathbb{F}_2$.
(a) Prove that $p(x)$ is irreducible over $\mathbb{F}_2$.
(b) Prove that $\mathbb{F}_2(\alpha)$ is a field with 8 elements.
(c) Prove that $\mathbb{F}_2(\alpha)$ is a Galois extension of $\mathbb{F}_2$ and compute $\mathrm{Gal}(\mathbb{F}_2(\alpha)/\mathbb{F}_2)$.

10. Suppose that $f \in \mathbb{Q}[x]$ with $\deg(f) = 5$. Let $K/\mathbb{Q}$ be the splitting field of f and suppose that $\mathrm{Gal}(K/\mathbb{Q}) = A_5$. Does there exist a field L between $\mathbb{Q}$ and K such that $[L : \mathbb{Q}] = 2$? Justify your answer.