PH.D. PRELIMINARY EXAMINATION IN ALGEBRA

JUNE 1996

Notation. In the following examination, G denotes a group, R denotes a ring. The symmetric group on n letters is denoted by S_n, the integers are denoted by $\mathbb{Z}$, and the rational numbers are denoted by $\mathbb{Q}$.

Problems.

(1) Find the order of the p-Sylow subgroup of S_n.

(2) Show that there is no simple group of order 225.

(3) Show that any group of order p^2q is solvable.

(4) Let G be a finite Abelian group. Let $\mathbb{QG}$ be the group algebra of G. Show that every irreducible $\mathbb{QG}$-module is isomorphic to a finite field extension of $\mathbb{Q}$.

(5) Show that for each prime p, the cyclotomic polynomial

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = \frac{x^p - 1}{x - 1}$$

is irreducible in $\mathbb{Z}[x]$.

(6) Let R be a (commutative) finite integral domain, show that R is a field.

(7) Let R be a commutative Noetherian ring. Show that $R[x]$ is a Noetherian ring.

(8) Let I be an ideal in the intersection of all maximal ideals of the ring R, and let M be a finitely generated R-module with $M = IM$. Show that M is the zero module.

(9) Give an example of a ring with a projective module that is not a free module.

(10) What is the Galois group of $x^4 - 2$ over $\mathbb{Q}$?.