Algebra Preliminary Examination

May 2016

Instructions:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital module".
- This exam has two pages.

1. Let R be a commutative ring with identity $1 \neq 0$. Let Q be a proper ideal of R. We say that Q is *primary* if whenever $ab \in Q$ and $a \notin Q$ then $b^n \in Q$ for some positive integer n. Also, recall that an element $x \in R$ is said to be *nilpotent* if $x^m = 0$ for some positive integer m.

(a) Prove that a prime ideal of R is primary.

(b) Prove that an ideal Q of R is primary if and only if every zero divisor in R/Q is a nilpotent element of R/Q.

(c) Prove that if Q is a primary ideal, then $\sqrt{Q} := \{r \in R \mid r^n \in Q \text{ for some } n \in \mathbb{Z}^+\}$ is a prime ideal.

2. Let I and J be ideals in a commutative ring R such that $I + J = R$. Let M and N be R-modules such that $IM = 0 = JN$. Prove that $M \otimes_R N = 0$.

3. Let V be a finite dimensional vector space and $\varphi : V \to V$ be an *idempotent* linear transformation (i.e., $\varphi^2 = \varphi$). Prove that $V = \text{image}(\varphi) \oplus \ker(\varphi)$.

4. Let R be an integral domain such that every R-module is flat. Prove that R is a field.

5. Let R be a non-zero commutative ring and $S \subseteq R$ be a multiplicatively closed subset. Given an R-module homomorphism $h : M \to N$, we define $S^{-1}h : S^{-1}M \to S^{-1}N$ by $m/s \mapsto h(m)/s$. You may assume that $S^{-1}h$ is an $S^{-1}R$-module homomorphism.

(a) Let $g : M \to N$ and $f : L \to M$ be two R-module homomorphisms. Prove that

$$S^{-1}(g \circ f) = (S^{-1}g) \circ (S^{-1}f).$$

(b) Suppose that the sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact at M. Prove that the sequence

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact at $S^{-1}M$.

6. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V over the field F. Prove that

$$\dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2).$$

7. Let p be a prime number and let S_p be the permutation group on the set $A = \{1, 2, ..., p\}$.

(a) Prove that if $\sigma \in S_p$ is a permutation of order p, then σ is a p-cycle.

(b) Let G be a subgroup of S_p. Prove that if the group action $G \times A \to A$ given by $\sigma \cdot k = \sigma(k)$ is transitive, then G contains a p-cycle.

8. Prove that every group of order 105 is solvable.

9. Let p be a prime and let $\mathbb{F}_p$ be the finite field with p elements. Suppose that $g(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial with $\deg(g) = d$. Prove that if d divides n, then $g(x)$ divides $x^{p^n} - x$ in $\mathbb{F}_p[x]$.

10. Let ω_8 be a primitive 8^{th} root of unity. Exhibit (with proof) the complete subfield lattice of $\mathbb{Q}(\omega_8)$.