Algebra Preliminary Examination

September 2013

Directions: Show all work for full credit. For this exam, R always denotes a commutative ring with identity and M denotes a unital R-module. Good luck and just do the best you can.

1. How many elements of order 7 must be in a *simple* group of order 168? Justify your answer.

2. Let S_n denote the group of permutations on the set $\{1, 2, ..., n\}$ and let A_n denote the subgroup of all even permutations. Prove that if H is a subgroup of S_n such that $H \nsubseteq A_n$, then exactly half of the permutations of H are even.

3. Consider the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.
(a) Prove that $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian ring.
(b) Is $\mathbb{Z}[\sqrt{-5}]$ a UFD? Briefly justify your answer.

4. For each prime ideal P of a ring R, define $\iota_P : R \to R_P$ to be the natural ring homomorphism given by $\iota_P(r) = \frac{r}{1}$.
(a) Prove that r is a unit in R if and only if $\iota_P(r)$ is a unit in R_P for each prime ideal P of R.
(b) Prove that $r = 0$ if and only if $\iota_P(r) = 0$ for each prime ideal P of R.

5. Let M be an R-module and let and let I be an ideal of the ring R. Prove that there exists an R-module isomorphism

$$R/I \otimes_R M \simeq M/IM.$$

6. An R-module M is called *finitely presented* if there exists an exact sequence of R-modules $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ for some positive integers m, n. Prove that every finitely generated projective R-module is finitely presented.

7. Let $R \subset T$ be an extension of commutative rings (sharing the same 1). An element $t \in T$ is said to be *integral* over R if there exists a monic polynomial $f \in R[x]$ such that $f(t) = 0$. If $\alpha \in T$, prove that α is integral over R if and only if the ring $R[\alpha]$ is finitely generated as an R-module.

8. Let V be a finite diminsional vector space over the field $\mathbb{Q}$ of rational numbers and let $\theta \in \mathrm{Hom}(V, V)$ be an invertible linear transformation. Prove that if θ satisfies the relation $\theta^{-1} = \theta^2 + \theta$, then 3 divides $\dim(V)$.

9. Suppose that $p(x) \in F[x]$ is irreducible with $\deg(p) = n$ and suppose that K/F is a finite field extension with $[K : F] = m$. Prove that if $\gcd(m, n) = 1$, then $p(x)$ is irreducible over K.

10. Determine all subfields of $\mathbb{Q}(i, \sqrt{2})$ and prove that your determination is complete.