Algebra Prelim September 2012

In this exam, the term "ring" is short for "commutative ring with identity" and "module" means "unital module". Let R be a ring.

Full credit will only be given for solutions that are completely justified.

1. Show that the polynomial $x^2 + y^2 - 1$ is irreducible in $\mathbb{R}[x, y]$.

2. Let p be a prime number. Show that an element in the symmetric group S_n has order p if and only if it is a product of commuting p-cycles.

3. Is the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$ diagonalizable over $\mathbb{C}$?

4. Let M be an R-module, and prove that the following conditions are equivalent:

   (i) $M = 0$.

   (ii) For every multiplicatively closed subset $U \subseteq R$, we have $U^{-1}M = 0$,

   (iii) For every prime ideal $P \subset R$, we have $M_P = 0$.

   (iv) For every maximal ideal $\mathfrak{m} \subset R$, we have $M_{\mathfrak{m}} = 0$.

5. Give an example of a finite normal field extension that is not Galois.

6. An R-module $M \neq 0$ is *simple* if its only submodules are 0 and M. Let M be a simple R-module. Prove that there is a unique maximal ideal $\mathfrak{m} \subset R$ such that $M \cong R/\mathfrak{m}$.

7. Prove that the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective. Is it projective?

8. Let $U \subseteq R$ be multiplicatively closed, and let $I \subseteq R$ be an ideal that is maximal among all ideals J such that $J \cap U = \emptyset$. Prove that I is prime.

9. Let $G = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ denote the group of invertible 2×2 matrices over the field $\mathbb{Z}/3\mathbb{Z}$. List the prime numbers p such that G has a non-trivial p-subgroup.

10. Let G be a finite abelian group and H a subgroup of G. Show that G has a subgroup isomorphic to G/H.