ALGEBRA PRELIMINARY EXAMINATION

AUGUST 2006

Abstract. In this examination all fields are commutative. All rings contain 1 and all modules are unitary. The rational numbers are denoted by $\mathbb{Q}$.

(1) Show that any group of order p^2q (where p and q are distinct prime integers) is the semidirect product of two abelian groups.

(2) Show that there is no simple group of order 392.

(3) Find (up to isomorphism type) all subgroups of S_5 of order 20.

(4) Let R be a commutative Noetherian ring. Show that the nilradical (that is, the set of nilpotent elements of R) is a nilpotent ideal.

(5) Let $R \subseteq T$ be commutative rings with identity and let P be a projective $R-$module. Show that $P \otimes_R T$ is a projective $T-$module.

(6) Let D be a PID with quotient field K. Show that any overring (that is, a ring contained between D and K) of D is of the form D_S for some multiplicatively closed subset $S \subseteq D$.

(7) Let M be a free $\mathbb{Z}-$module that is also injective. Show that $M = 0$.

(8) Suppose that $f(x) \in \mathbb{Q}[x]$ is a fifth degree, irreducible polynomial with Galois group A_5. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Does there exist a field E ($\mathbb{Q} \subseteq E \subseteq K$) such that $[E : \mathbb{Q}] = 2$?

(9) An $R-$module P is said to be projective if given any surjective $R-$module homomorphism $f : A \longrightarrow B$ and $R-$module homomorphism $g : P \longrightarrow B$, then there is an $R-$module homomorphism $h : P \longrightarrow A$ such that $fh = g$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ A \arrow[r, "f"'] & B \arrow[r] & 0 \end{tikzcd}$$

Show that any free $R-$module is projective.

(10) Let $I \subseteq R$ be an ideal. Show that the following conditions are equivalent.
a) I is radical and primary.
b) I is prime.