ALGEBRA PRELIMINARY EXAMINATION

JANUARY 2012

(1) Show that no group of order 80 is simple.
(2) Show that every finite group of order p^n (where p is a positive prime integer) has a nontrivial center.
(3) Let G be a group with center Z. Show that if the index of Z in G is n, then G has at most n^2 distinct commutators.
(4) Let R be an integral domain and $I \subsetneq R$ a proper ideal. Show that if R/I is a projective $R-$module, then $I = 0$.
(5) Let R be a commutative ring with identity, and let P be an $R-$module. Show that P is a projective $R-$module if and only if given any $R-$module epimorphism $g : B \longrightarrow C$, the induced $R-$module homomorphism

$$\overline{g} : \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C)$$

is onto.
(6) Let R be a commutative ring with identity and $n \in \mathbb{N}$. Suppose that $I \subseteq R$ be an ideal that cannot be generated with n elements. Let X_n be the collection of ideals of R that cannot be generated by n elements. Show that the set X_n has a maximal element.
(7) We say an integral domain R with quotient field K is a *valuation domain* if given nonzero $a, b \in R$, then either a divides b or b divides a. Show that if R is a valuation domain with quotient field K and T is a domain such that $R \subseteq T \subseteq K$, then T is a valuation domain.
(8) Find the Galois group of the extension $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$.
(9) Let $\mathbb{F}_2$ be the field of two elements. Let $\mathbb{K}$ be the extension of $\mathbb{F}_2$ generated by adjoining all of the roots of the polynomial $x^5 + x + 1$. Find the Galois group of $\mathbb{K}$ over $\mathbb{F}_2$.
(10) We say that an integral domain R is ACCP (ascending chain condition on principal ideals) if there is no infinitely ascending chain of principal ideals. Show that if every ideal of R is finitely generated, then R is ACCP.