ALGEBRA PRELIMINARY EXAMINATION

SEPTEMBER 2011

(1) Let $n \geq 3$ be a natural number.
  (a) Show that A_n has a subgroup of index n.
  (b) Show that if $n \geq 5$ and $1 < k < n$, then A_n has no subgroup of index k.

(2) Show that there is no simple group of order 96.

(3) Let G be a finite group ($|G| > 4$) that is generated by two elements of order 2. Show that G is dihedral.

(4) Let R be a commutative ring with identity. Show that if I is an ideal that is maximal with respect to being non-principal, then I is prime.

(5) Let R be a commutative ring with identity and P an $R-$module. Show that the following conditions are equivalent.
  a) P is projective.
  b) Given the short exact sequence of $R-$modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

  the induced sequence of $R-$modules

$$0 \longrightarrow Hom_R(P, A) \xrightarrow{\overline{f}} Hom_R(P, B) \xrightarrow{\overline{g}} Hom_R(P, C) \longrightarrow 0$$

is also exact.

(6) Let R be a commutative ring with identity. Show that if P, Q are projective $R-$modules, then so is $P \otimes_R Q$.

(7) Let R be a commutative ring with identity. Recall that the Jacobson radical of R ($J(R)$) is the intersection of all maximal ideals of R. Show that $x \in J(R)$ if and only if $1 + rx$ is a unit in R for all $r \in R$.

(8) Give an example of a field extension $K \subseteq F$ such that $[F : K] = 2$, yet F is not Galois over K.

(9) Show that if $x^3 + Ax^2 + Bx + C$ is irreducible over $\mathbb{Q}$ and has Galois group $\mathbb{Z}/3\mathbb{Z}$ then $A^2 > 3B$.

(10) Show that R is a UFD with the property that every nonzero prime ideal of R is maximal, then R is a PID.