Preliminary Examination (Math 720)
21 January 2022

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\varphi : R \to S$ is a ring homomorphism, we also assume $\varphi(1_R) = 1_S$

1. Let R be a commutative ring and I, J, K ideals of R. Prove that if $I + J = I + K = R$, then $I + JK = R$.

2. Prove that the ring $\mathbb{Z}[i]/I$ is finite for every nonzero ideal I of $\mathbb{Z}[i]$.

3. Let $\mathbb{Z}$ be the ring of integers and let X be an indeterminate over $\mathbb{Z}$. How many elements does the ring $\mathbb{Z}[X]/(X^2 - 3, 2X + 4)$ have? Justify your answer.

4. Let S be an integral domain, and let R be a subring of S such that S is finitely generated as an R-module. Assume that R is a field. Prove that S is a field.

5. Let $I = (2, X)$ be the ideal generated by 2 and X in the ring $R = \mathbb{Z}[X]$. Show that the element $2 \otimes 2 + X \otimes X$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.

6. Let G be the additive abelian group $\mathbb{Z} \times \mathbb{Z}$. Prove that the quotient group $G/\langle(1, 1)\rangle$ is an infinite cyclic group.

7. Let R be an integral domain. Recall that an R-module N is called torsion free if $\mathrm{Tor}(N) = (0)$ where $\mathrm{Tor}(N) := \{x \in N \mid rx = 0 \text{ for some } r \in R \setminus \{0\}\}$.

Prove that if M is a projective R-module then M is torsion free.