Preliminary Examination (Math 720)
18 August 2021

INSTRUCTIONS:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\varphi : R \to S$ is a ring homomorphism, we also assume $\varphi(1_R) = 1_S$.

1. Let $R = \mathbb{Z}[X]$ and the ideal $I = (6, X^2 + X + 1)$ of R. Find all the ideals of R that contain I. (Give an explicit list of generators for each ideal.) Which of these ideals are prime? Which of these ideals are maximal?

2. Consider the subring $\mathbb{Z}[\sqrt{10}]$ of the field of real numbers. Show that 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{10}]$.

3. Let $n \geq 1$ be an integer and consider the real vector space $V = \{f \in \mathbb{R}[X] \mid \deg(f) \leq n\}$. Let $T : V \to V$ be the linear transformation given by $T(f) = f'$, where f' is the derivative of f. Find the characteristic polynomial, minimal polynomial, invariant factors, elementary divisors, rational canonical form, and Jordan canonical form of the linear transformation T.

4. Prove that $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = (0)$

5. Let R be a commutative ring.

(a) Assume that I and J are ideals of R such that $I \cap J = (0)$ and $I + J = R$. Prove that there exists $e \in R$ with $e^2 = e$ such that $I = (e)$.

(b) Show that a principal ideal aR in R is projective as an R-module if and only if $\text{ann}(a) := \{r \in R \mid ra = 0\}$ (the annihilator of a) is of the form eR where $e = e^2$.

6. Let R be a commutative ring. Assume that R has a unique maximal ideal. Prove that 0 and 1 are the only idempotents in R. (An element $x \in R$ is said to be idempotent if $x^2 = x$.)

7. Suppose I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements.