# Algebra Preliminary Examination

August 2020

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1_R) = 1_S$.
- This exam has two pages.

## A. Rings, Modules, and Linear Algebra

1. Consider three properties that a ring $R$ might have:

$$(P_1)\ \text{Noetherian} \quad (P_2)\ \text{PID} \quad (P_3)\ \text{UFD}.$$

   (a) For which $n \in \{1, 2, 3\}$ is it true that if $R$ has property $P_n$ then $R[x]$ has property $P_n$?

   (b) For which $n \in \{1, 2, 3\}$ is it true that if $R[x]$ has property $P_n$ then $R$ has property $P_n$?

   If the implication is true, supply a short proof or just cite a well-known theorem to justify your answer. If it is false, exhibit a counterexample.

2. Let $\Sigma$ be the set of all proper ideals of a ring $R$ that consist only of zero-divisors.

   (a) Prove that $\Sigma$ has maximal elements with respect to inclusion.

   (b) Prove that every maximal element of $\Sigma$ is a prime ideal.

3. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for the $\mathbb{R}$-vector space $V = \mathbb{R}^2$. Prove that $(\mathbf{e}_1 \otimes \mathbf{e}_2) + (\mathbf{e}_2 \otimes \mathbf{e}_1)$ cannot be written as a simple tensor in $V \otimes_{\mathbb{R}} V$.

4. Prove that the quotient module $\mathbb{Q}/\mathbb{Z}$ is not finitely generated as a $\mathbb{Z}$-module.

5. Let $U, V, W$ be vector spaces over the field $F$. Prove that if $S : U \to V$ and $T : V \to W$ are linear transformations, then $\dim(\ker(T \circ S)) \leq \dim(\ker T) + \dim(\ker S)$.

6. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Find the rational canonical form for the matrix $A$ over $\mathbb{Q}$.

   (b) Find the Jordan canonical form for the matrix $A$ over $\mathbb{C}$.

**B. Homological Algebra**

7. Let $R$ be a commutative ring, $M$ an $R$-module and $x \in R$ an element that is both $R$-regular and $M$-regular. Prove that

$$\operatorname{pd}_{R/xR} M/xM \leq \operatorname{pd}_R M.$$

8. For a field $k$, let $T = k[X,Y]/(XY)$ and denote $x = \overline{X} \in T$. Consider $k \cong T/(x,y)T$ as a $T$-module.

   (a) Compute $\operatorname{Tor}_n^T(k, T/xT)$ for every $n \geq 0$.

   (b) What is $\operatorname{pd}_T(T/xT)$?

9. Let $R$ be a principal ideal domain and let $a, b \in R \setminus \{0\}$. Compute $\operatorname{Ext}_R^n(R/aR, R/bR)$ for all $n \geq 0$.

10. Let $R$ be a commutative noetherian ring and let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated $R$-modules. Let $I$ be an ideal of $R$ contained the Jacobson radical $\operatorname{Jac}(R)$. Prove that:

$$\operatorname{depth}_I B \geq \min\{\operatorname{depth}_I A, \operatorname{depth}_I C\}.$$

Moreover, if the sequence is split exact, we have equality, i.e.,

$$\operatorname{depth}_I A \oplus C = \min\{\operatorname{depth}_I A, \operatorname{depth}_I C\}.$$