Algebra Preliminary Examination
August 2017

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1) = 1$.
- This exam has two pages.

1. Let R be a ring.
   (a) Prove or disprove: If the polynomial ring $R[x]$ is a PID, then R is a field.
   (b) Prove or disprove: If R is a field, then the polynomial ring $R[x, y]$ is a Euclidean domain.
2. Let S be a multiplicatively closed subset of a ring R. Define the *saturation* of S to be the set $\bar{S} = \{v \in R : v \text{ divides } s \text{ for some } s \in S\}$. Prove that $\bar{S}$ is multiplicatively closed and that there exists a ring isomorphism $S^{-1}R \simeq \bar{S}^{-1}R$.
3. Let R be an integral domain and let M be an R-module such that $\mathrm{Hom}_R(M, R/I) = 0$ for all nonzero ideals $I \leq R$. Prove that $\mathrm{Hom}_R(M, R) = 0$.
4. Recall that an R-module M is called divisible if for every nonzero $r \in R$, one has the equality $rM = M$. Let R be a PID and let Q be an R-module. Prove that Q is divisible if and only if it is an injective R-module.
5. Consider the $\mathbb{R}$-vector space of all differentiable real-valued functions of a single variable and let V be the subspace spanned by the ordered basis $B = \{t\sin(t), t\cos(t), \sin(t), \cos(t)\}$. Let $\Delta : V \to V$ be the usual differential operator given by $\Delta(\varphi(t)) = \varphi'(t)$.
   (a) Find the matrix A of the operator Δ relative to the basis B.
   (b) Find the rational canonical form (over $\mathbb{R}$) of the matrix A.
   (c) Find the Jordan canonical form (over $\mathbb{C}$) of A.

6. Let V be an n-dimensional vector space over the field F and let $T : V \to V$ be a linear transformation.

   (a) Prove that $\ker(T^n) = \ker(T^{n+k})$ and $\mathrm{Im}(T^n) = \mathrm{Im}(T^{n+k})$ for all $k \geq 0$.

   (b) Prove that $V = \ker(T^n) \oplus \mathrm{Im}(T^n)$.

7. Prove that if σ, τ are elements of order $2, 3$ (resp.) in A_4, then $\langle \sigma, \tau \rangle = A_4$.

8. Let G be a group and recall that centralizer and normalizer of any subset $P \subseteq G$ are the sets $C_G(P) = \{g \in G : gp = pg \text{ for all } p \in P\}$ and $N_G(P) = \{g \in G : gPg^{-1} = P\}$ respectively. If G is a group of order $660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ and P is a Sylow 11-subgroup such that $C_G(P) = P$, determine the cardinality $|N_G(P)|$ of the normalizer of P in G.

9. Let $F \subseteq K$ be an extension of fields such that $[K : F] < \infty$ and let $f(x) \in F[x]$ with $\deg(f) = p$ prime. Prove that if $f(x)$ is irreducible in $F[x]$ and reducible in $K[x]$, then p divides $[K : F]$.

10. Let K be a splitting field of the polynomial $x^8 - 2 \in \mathbb{Q}[x]$. Exhibit (with proof) all intermediate fields between $F = \mathbb{Q}(i, \sqrt{2})$ and K.