Algebra Preliminary Examination
January 2020

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- For this exam you have two options:
  (i) Submit solutions to questions from part A and from part B.
  (ii) Submit solutions to questions from part A and from part C.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1_R) = 1_S$.
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let $R = \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients and let I be the 2-generated ideal $(2, x^3 + 1)$. **Prove or disprove** each statement.
   (a) I is a prime ideal of R.
   (b) I is a maximal ideal of R
2. Let R be the subring $\mathbb{Z}[2i] = \{a + 2bi : a, b \in \mathbb{Z}\}$ of the ring $\mathbb{Z}[i]$ of Gaussian integers.
   (a) Prove that the elements 2 and $2i$ are irreducible in R.
   (b) Prove that $\mathbb{Z}[2i]$ is *not* a UFD.
3. Consider the short exact sequence $0 \to L \to M \to N \to 0$ of R-modules. Suppose that N is a projective R-module. Prove that M is projective if and only if L is projective.
4. Let R be an integral domain such that every one of its R-modules is free. Prove that R is a PID.
5. Let W, X be subspaces of the F-vector space V. Prove that if $V = W + X$ and $\dim(V) = \dim(W) + \dim(X)$, then $V = W \oplus X$.
6. Let V be an n-dimensional vector space over the field $\mathbb{Q}$ of rational numbers. and let $T \in \mathrm{End}_{\mathbb{Q}}(V)$ be a linear transformation.
   (a) Prove that if T satisfies $T^2 = T$, then it is diagonalizable.
   (b) Up to similarity, how many such $\mathbb{Q}$-endomorphisms of V are there? Justify your answer.

B. Groups, Fields, and Galois Theory (option 1)

1. Consider the group S_4 of all permutations on the set $\{1, 2, 3, 4\}$ and let A_4 be the alternating group of all even permutations.

   (a) Prove that A_4 has no subgroup of order 6.

   (b) Prove that A_4 is the only subgroup of S_4 of order 12.

2. Classify all groups of order $175 = 5^2 \cdot 7$.

3. Let $F \subseteq K$ be an extension of fields with $u \in K$ transcendental over F. Prove that every element of $F(u) - F$ is transcendental over F.

4. Give an example of a field tower $F \subseteq L \subseteq K$ such that $F \subseteq L$ and $L \subseteq K$ are normal extensions, but $F \subseteq K$ is not normal.

C. Homological Algebra (option 2)

1. Let A, B be two finitely generated $\mathbb{Z}$-modules. Prove that

$$\mathrm{Tor}_2^{\mathbb{Z}}(A, B) = 0.$$

2. Let R be a principal ideal domain and M and N finitely generated torsion R-modules. Prove that there exists an isomorphism of R-modules $\mathrm{Tor}_1^R(M, N) \cong M \otimes_R N$.

3. Let R be a commutative ring, M an R-module, $\underline{x} = x_1, \ldots, x_n$ an M-regular sequence. Denote $I = (x_1, \ldots, x_n) \subseteq R$. Assume that we have an exact sequence of R-modules

$$N_2 \to N_1 \to N_0 \to M \to 0.$$

   Prove that the induced sequence

$$N_2/IN_2 \to N_1/IN_1 \to N_0/IN_0 \to M/IM \to 0$$

   is exact.

4. Let I, J be ideals in a commutative ring R. Prove that we have the following isomorphisms of R-modules:

   (a) $\mathrm{Tor}_n^R(R/J, R/I) \cong \mathrm{Tor}_{n-2}^R(J, I)$ for $n > 2$.

   (b) $\mathrm{Tor}_2^R(R/J, R/I) \cong \mathrm{Ker}(J \otimes_R I \to JI)$.

   (c) $\mathrm{Tor}_1^R(R/J, R/I) \cong (J \cap I)/(JI)$.