Algebra Preliminary Examination
May 2017

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1) = 1$.
- This exam has two pages.

1. Is the ideal $(x^2 + 1, 11)$ maximal in the polynomial ring $\mathbb{Z}[x]$? Justify your answer.
2. Prove that the subring $\mathbb{Q}[x^2, x^3]$ consisting of all polynomials in $\mathbb{Q}[x]$ with zero linear term is *not* a UFD.
3. Prove that a finitely generated projective module M over a PID R is free.
4. Let V, W be finite dimensional vector spaces over a field F. If $\dim(V) = m$ and $\dim(W) = n$, what is $\dim(V \otimes_F W)$? Justify your answer. Feel free to use the usual calculus of tensor products.
5. Consider the matrix $A \in \mathcal{M}_3(\mathbb{Q})$ given by
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
   (a) Find the rational canonical form of A.
   (b) Is A diagonalizable? Justify your answer.
6. Let I be an ideal of the ring R and let M be an R-module generated by n elements. Prove that if $r \in R$ satisfies $rM \subseteq IM$, then there exists an element $y \in I$ such that $(r^n + y)M = 0$. **Hint:** Determinants.

7. Let $G = D_8$ be the dihedral group on the vertices of a square and let V_4 denote the Klein 4-group.

   (a) Prove that $G \simeq V_4 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$ for some group homomorphism $\theta : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(V_4)$.

   (b) Is it possible that the map $\theta : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(V_4)$ from part (a) is the trivial homomorphism? Justify your answer.

8. Prove that a group of order 96 must have a normal subgroup of order 16 or 32.

9. If $u = \sqrt{2} + \sqrt[3]{5}$, prove that $\mathbb{Q}(u) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ and find the degree of the minimal polynomial $m_{u,\mathbb{Q}}(x) \in \mathbb{Q}[x]$.

10. Let $\varphi_p(x) = 1 + x + ... + x^{p-1}$ where p is an odd prime and let K be the splitting field of φ_p over $\mathbb{Q}$. Prove that there exists a unique field L between $\mathbb{Q}$ and K such that $[K : L] = 2$.