Algebra Preliminary Examination

February 2015

• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Classify all groups of order 175. (Hint: first prove that any such group must be abelian.)

2. Let H be a (not necessarily normal) subgroup of the group G. Prove that if $[G : H] = n$, then $g^{n!} \in H$ for all $g \in G$.

3. Prove that the polynomial $X^5 - \sqrt[3]{2}$ is irreducible over the field $\mathbb{Q}(\sqrt[3]{2})$.

4. Let K be the splitting field of some irreducible polynomial $f(X) \in \mathbb{Q}[X]$. Prove that if $[K : \mathbb{Q}] = 102$, then there exists a field E with $\mathbb{Q} \subseteq E \subseteq K$ such that $\mathbb{Q} \subseteq E$ is a normal extension and $[E : \mathbb{Q}] = 6$.

5. Let A be a commutative ring, I an ideal of A, and $S \subseteq A$ a multiplicatively closed subset with $1 \in S$. Let $\phi : A \to S^{-1}A$ be the canonical ring homomorphism defined by $\phi(a) = a/1$ for all $a \in A$. For each $y \in A$, denote

$$(I : y) = \{x \in A \mid xy \in I\}.$$

Prove that

$$\phi^{-1}(\phi(I)S^{-1}A) = \bigcup_{s \in S} (I : s).$$

6. Let G be a finite abelian group with identity e. Prove that there exists an element $x \in G \setminus \{e\}$ such that $\operatorname{ord} y \mid \operatorname{ord} x$ for all $y \in G \setminus \{e\}$.

7. Let k be a field with $\operatorname{char} k \neq 2$, V a k-vector space, and $f \in \operatorname{End}_k(V)$ such that $f \circ f = 1_V$. Prove that

$$V_1 := \{x + f(x) \mid x \in V\} \text{ and } V_2 := \{x - f(x) \mid x \in V\}$$

are subspaces of V and $V = V_1 \oplus V_2$.

8. Prove that the polynomial $X^2Y + X^2 + Y$ is irreducible in $\mathbb{Q}[X, Y]$.

9. Let R be a commutative ring and M an R-module. We say that M is simple if the following two conditions are satisfied:

(a) $M \neq (0)$

(b) (0) and M are the only submodules of M.

Prove that M is simple if and only if there exists a maximal ideal $\mathfrak{m}$ of R such that M is isomorphic (as an R-module) to $R/\mathfrak{m}$.

10. Let I, J be ideals in a commutative ring R. Prove that there exists an isomorphism of R-modules

$$R/I \otimes_R R/J \cong R/(I + J).$$