Algebra Preliminary Examination

January 2011

• Begin each question on a new sheet of paper.
• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Find the minimal polynomial of $\sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.

2. Let $F \subseteq K$ be a field extension such that every irreducible polynomial in $F[X]$ remains irreducible in $K[X]$. Show F is algebraically closed in K.

3. Let G be the group of automorphisms of $\mathbb{Q}(\sqrt{5}, i)$.

(a) Find all the subgroups of G.

(b) Find all the subfields of $\mathbb{Q}(\sqrt{5}, i)$.

4. Show that every group of order 12 has a normal Sylow subgroup and hence is not simple.

5. Prove that every group of order 45 is abelian.

6. Let D be a commutative integral domain and F a subring of D. Assume that F is a field and D has finite dimension as a vector space over F. Prove D is a field.

7. (a) Prove that the ring $R = \mathbb{Z}[\sqrt{-2}]$ is Euclidean.

(b) Prove that that $R/(3 + 2\sqrt{-2})$ is a field with 17 elements.

8. Let M be a module over the integral domain D. A submodule N of M is said to be pure if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

If N is a direct summand of M, prove that N is pure in M.

9. Let R be a commutative ring, M an R-module, and N a submodule of M. Denote $i : N \to M$ the natural inclusion map.

(a) Show that if M/N is free, then the map

$$i^* : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(N, R)$$

is surjective, where $i^*(f) = f \circ i$.

(b) Give an example that shows that i^* need not be surjective if M/N is not free.

10. Give an example of a commutative ring A and an ideal I in A such that I cannot be generated by finitely many elements of A. (Make sure that you prove that your ideal cannot be generated by finitely many elements.)