Algebra Preliminary Examination

June 2010

• Begin each question on a new sheet of paper.
• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Let K be a field, $f \in K[X]$ a polynomial of degree n and L a splitting field of f. Prove that $[L : K]$ divides $n!$.

2. Let K be a field and let G be a finite subgroup of the multiplicative group $K^* = K \setminus \{0\}$. Prove that G must be cyclic.

3. (a) Let G be a finite group and H a subgroup of G of index n. Assume that H does not contain any non-trivial normal subgroups of G. Prove that G is isomorphic to a subgroup of S_n.

(b) Prove that there is no simple group of order 216.

4. (a) Show that the splitting field of $X^4 + 4X^2 + 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{-2 + \sqrt{2}})$.

(b) Compute the Galois group of the polynomial $X^4 + 4X^2 + 2 \in \mathbb{Q}[X]$.

(c) Find all the subfields of $\mathbb{Q}(\sqrt{-2 + \sqrt{2}})$.

5. Let R be a commutative ring and let M, N be R-submodules of an R-module L. Prove that if $M + N$ and $M \cap N$ are finitely generated, then so are M and N.

6. Let A be a commutative ring and L a free A-module of rank n. Let $x_1, \ldots, x_n \in L$.

(a) Assume that $x_1, \ldots, x_n$ generate L. Prove that $x_1, \ldots, x_n$ is a basis of L.

(b) If $x_1, \ldots, x_n$ are linearly independent, is it necessarily true that $x_1, ..., x_n$ form a basis of L? If yes, give a proof. If no, give a counterexample.

7. (a) Let H be a finitely generated subgroup of the abelian group $(\mathbb{Q}, +)$. Prove that H is cyclic.

(b) Prove that the abelian group $(\mathbb{Q}, +)$ is not finitely generated.

8. Let R be an integral domain. Denote by $\mathrm{Max}(R)$ the set of all maximal ideals of R. For each $m \in \mathrm{Max}(R)$ we denote by R_m the localization of R at the maximal ideal m. Note that each R_m is a subring of the fraction field of R. Prove that $R = \bigcap_{m \in \mathrm{Max}(R)} R_m$.

9. (a) Prove that the ring $R = \mathbb{Z}[X]/(2, X^2 + 1)$ has four elements.

(b) Prove that R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

10. Let $R = \mathbb{Z}[\sqrt{-5}]$.

(a) Prove that the ideal $I = (2, 1 + \sqrt{-5})$ is not principal.

(b) Prove that the product of two non-principal ideals in R can be principal.