Algebra Preliminary Examination

May 2007

• Begin each question on a new sheet of paper.

• In answering any part of a question, you may assume the results in previous parts, even if you have not solved them.

All rings have identity and all modules are unitary.

1. (a) Let G be a group and H a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Prove that there exists a homomorphism $\phi : G \to S$ whose kernel lies in H and contains every normal subgroup of G that is contained in H.

(b) Prove that a group with 80 elements is not simple.

(c) Prove that a simple group with 60 elements has a subgroup of order 6 and a subgroup of order 10.

2. (a) Find an element of maximal order in S_5.

(b) Describe (up to isomorphism) all the Sylow p-subgroups of S_5.

(c) For each p, indicate the number of Sylow p-subgroups.

3. Let p and q be distinct prime numbers. Prove that every group of order p^2q is solvable.

4. Let p be a prime number, $L = \mathbb{F}_p(X, Y)$ and $K = \mathbb{F}_p(X^p, Y^p)$ ($\mathbb{F}_p$ is the field with p elements). For each positive integer n, consider the element $\alpha_n = X + YX^{p^n} \in L$ and the subfield $E_n = K(\alpha_n) \subseteq L$. Prove the following statements:

(a) $[L : K] = p^2$ and $[E_n : K] = p$ for all n.

(b) $E_n \neq E_m$ if $n \neq m$.

5. Let $f(x) = x^4 + 4x^2 + 2$.

(a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) Let L be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine the Galois group $G(L/\mathbb{Q})$.

6. Let R be a commutative ring and let $\mathfrak{m}$ and $\mathfrak{n}$ be two distinct maximal ideals of R. Prove that $\mathfrak{m} \cap \mathfrak{n} = \mathfrak{m}\mathfrak{n}$.

7. Let R be a ring. Prove that the following are equivalent:

(a) Every R-module is injective.

(b) Every R-module is projective.

8. Let R be a Unique Factorization Domain. Choose an irreducible element $p \in R$, and define the localization at p as the ring of fractions $R_p = D^{-1}R$ with respect to the multiplicative set $D = R - (p)$. Show that R_p is a Principal Ideal Domain.

9. Let R be a commutative ring and I an ideal of R. For an R-module M, we denote $(0 :_M I) = \{x \in M \mid xI = 0\}$. Prove that $\mathrm{Hom}_R(R/I, M) \cong (0 :_M I)$.

10. Let R be a noetherian integral domain. Prove that any non-zero and non-unit element in R is a product of irreducible elements.