Algebra Preliminary Examination
May 2018

INSTRUCTIONS:
• Write your student ID number at the top of each page of your exam solution.
• Write only on the front page of your solution sheets.
• Start each question on a new sheet of paper. Each question is worth 10 points.
• For this exam, you have **two options**:

- Option 1: Submit solutions to questions from Part A and from Part B.
- Option 2: Submit solutions to questions from Part A and from Part C.

• In answering any part of a question, you may assume the results in previous parts.
• To receive full credit, answers must be justified.
• In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\phi : R \to S$ is a ring homomorphism, we also assume $\phi(1_R) = 1_S$.
• This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. (a) Prove that $\mathbb{Z}[X]/(X^2+1)$ is isomorphic to a subring of the field of complex numbers.

(b) Prove that $\mathbb{Z}[X]/(X^2-6, X^2+1)$ is a field.

2. Let R be a commutative ring. Let I and J be ideals of R such that $I + J = R$. Prove that there exists an isomorphism of rings $R/(I \cap J) \cong R/I \times R/J$.

3. Let R be an integral domain. Assume that every element of R is a product of finitely many prime elements of R.

(a) Prove that if a is an irreducible element of R, then a is a prime element of R.

(b) Prove that R is a unique factorization domain.

4. Let R be a commutative ring and L an R-module. Let M, N be R-submodules of L such that $M \cap N$ and $M + N$ are finitely generated. Prove that M and N are finitely generated.

5. Let R be a commutative ring and M an R-module generated by $y_1, \ldots, y_n \in M$.

(a) Let $\varphi : R \to M^n$ defined by $\varphi(a) = (ay_1, \ldots, ay_n)$. Prove that φ is an R-module homomorphism.

(b) Assume that M is a noetherian R-module. Prove that $R/\operatorname{Ann}_R(M)$ is a noetherian ring. (Recall that $\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\}$.)

(c) If M is noetherian, is it necessarily true that the ring R is noetherian? (Justify your answer.)

6. If F is a field, V is a finite dimensional F-vector space, and $T : V \to V$ is an F-linear map, we denote by V_T the F-vector space V with the $F[X]$-module structure induced by $X \cdot v = T(v)$ for all $v \in V$.

Let F be a field, and V, W finite dimensional F-vector spaces. Let $T : V \to V$ and $S : W \to W$ be F-linear maps. Prove that

$$\mathrm{Hom}_{F[X]}(V_T, W_S) = \{U \in \mathrm{Hom}_F(V, W) \mid U \circ T = S \circ U\}$$

B. Groups, Fields, and Galois Theory (option 1)

1. Classify all groups of order $2p$ where p is an odd prime.

2. Write down all Sylow subgroups of A_4. Justify your answers.

3. Let $F \subseteq K$ be an extension of fields such that $\mathrm{char}(F) = p$ is prime. Fix any algebraic element $u \in K$ and let $m_{u,F}(X) \in F[X]$ be its minimal polynomial. Prove that $m_{u,F}(X)$ is a separable polynomial if and only if $F(u) = F(u^p)$.

4. (a) Determine the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}(\sqrt{105}))$.

(b) Exhibit (with proof) the complete lattice of subfields between $\mathbb{Q}(\sqrt{105})$ and $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$.

C. Homological Algebra (option 2)

1. Let m, n be integers and let $d = \gcd(a, b)$. Prove that

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}.$$

2. Let R be a commutative ring and let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of R-modules. Prove that

$$\mathrm{pd}_R\, M \leq \sup\{\mathrm{pd}_R\, M_1, \mathrm{pd}_R\, M_2\}.$$

Moreover, if $\mathrm{pd}_R\, M < \sup\{\mathrm{pd}_R\, M_1, \mathrm{pd}_R\, M_2\}$, prove that

$$\mathrm{pd}_R\, M_2 = \mathrm{pd}_R\, M_1 + 1.$$

3. Let R be a commutative ring and M an R-module. Prove that if $x \in R$ is a non-zerodivisor on both R and M, then $\mathrm{Tor}_i^R(M, R/xR) = 0$ for $i \geq 1$.

4. Let A be a $\mathbb{Z}$-module and such that $nA = 0$ for some non-zero integer n.

(a) Prove that $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) = 0$.

(b) Prove that $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/n\mathbb{Z})$.