Algebra Preliminary Examination
September 2010

• Begin each question on a new sheet of paper.
• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Let G be a group and let H be a subgroup of the center $Z(G)$. Assume that G/H is cyclic. Prove that G is abelian.

2. Let X be a subgroup of $(\mathbb{Q}, +)$ such that $X + \mathbb{Z} = \mathbb{Q}$. Prove that $X = \mathbb{Q}$.

3. Let G be a group with a non-trivial subgroup H of index $r > 1$. Prove the following:

(a) If G is simple, then $|G|$ divides $r!$.

(b) If $r \in \{2, 3, 4\}$, then G cannot be simple.

4. Prove that a group of order 105 has a subgroup of order 35.

5. Let R be a commutative ring and A, B be R-modules. Prove that if $f : A \to B$ and $g : B \to A$ are R-module homomorphisms such that $g \circ f = 1_A$, then $B \cong \operatorname{Im} f \oplus \operatorname{Ker} g$.

6. Let R be a commutative ring and I, J ideals of R with $I \cap J = 0$. If R/I and R/J are noetherian rings, then R is a noetherian ring.

7. Let R be a ring with identity 1 (but not necessarily commutative). Prove that the following are equivalent:

(a) R is a field.

(b) For every $a \in R \setminus \{1\}$ there exists $b \in R$ such that $a + b = ab$.

(c) For every $a \in R \setminus \{1\}$ there exists $b \in R$ such that $a + b = ba$.

8. Prove that $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$ is a Galois extension of $\mathbb{Q}$ of degree 4 with cyclic Galois group.

9. Let d, n be positive integers. Prove that d divides n if and only if $X^d - 1$ divides $X^n - 1$.

10. Let G be a finite abelian group of order n.

(a) Prove that for any divisor q of n there exists a subgroup of order q.

(b) Prove that there exists an element $x \in G$ such that $\operatorname{ord}(y)$ divides $\operatorname{ord}(x)$ for all $y \in G$.