Analysis Premilinary Exam
January 24, 2009

1. Let $(X, \mathcal{M}, \mu)$ and, for every $n \geq 1$, let $f_n : X \to \mathbb{R}$ be measurable functions. Prove that $\limsup_{n\to\infty} f_n$ is measurable.

2. Let f be an integrable function on $\mathbb{R}$ with Lebesgue measure m. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\displaystyle\int_E |f|\,dm < \epsilon$ whenever $m(E) < \delta$.

3. Let $h_n(x) = n\sin\left(\frac{x}{n}\right), 0 \leq x \leq \pi$, $n \geq 1$. Find
$$\lim_{n\to\infty}\int_0^{\pi} h_n(x)\,dx.$$
Hint: Use properties of the function $\dfrac{\sin y}{y}$.

4. Let $X = Y = [0,1]$, with Lebesgue measure m on X and the counting measure ν on Y. In the product space $(X\times Y, m\otimes\nu)$ we consider the set $V = \{(x,y) : x = y\}$.
   (a) Show that the iterated integrals of χ_V have different values.
   (b) Does the result in (a) contradict Tonelli's Theorem? (Justify.)

5. Let $p \in (1,\infty)$ and $f \in L^p[0,1]$. Show that
$$\lim_{y\to 0^+} y^{\frac{1-p}{p}}\int_0^y f(x)\,dx = 0.$$

6. Let μ be a signed measure on $[0,\infty)$ given by $\mu(\Lambda) = \displaystyle\sum_{n\in\Lambda\cap\mathbb{N}}\frac{(-1)^n}{n^2}$, where the sums are taken in the natural order on $\mathbb{N}$.
   (a) Show that μ can be decomposed as $\mu = \nu - \tau$, where ν and τ are positive measures.
   (b) Explain why the decomposition in (a) does not work for the function $\gamma(\Lambda) = \displaystyle\sum_{n\in\Lambda\cap\mathbb{N}}\frac{(-1)^n}{n}$