Analysis Qualifying **January 2018**

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis

Lebesgue measure is denoted by m, the corresponding outer measure is denoted by m^*.

1. Let S be the collection of all subsets of $[0,1)$ which can be written as a finite union of intervals of the form $[a,b) \subseteq [0,1)$. Show that S is an algebra of sets, but is not a σ-algebra.

2. Let $A \subseteq \mathbb{R}^n$ such that $m^*(A) = 0$. Show that A is Lebesgue measurable.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Show that f is Borel measurable.

4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $f \in L^1(\mu)$. Prove that
$$\int_X |f|\, d\mu = \int_0^\infty \mu(\{x \in X : |f(x)| > t\})\, dt.$$
Hint: Write the right-hand side as a double integral and use Tonelli.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of integrable functions such that
$$\sum_{n=1}^\infty \int |f_n| d\mu < \infty.$$
   (a) Show that the series
   $$\sum_{n=1}^\infty f_n(s)$$
   is absolutely convergent for almost every $s \in X$.

   (b) Let
   $$f(s) = \sum_{n=1}^\infty f_n(s)$$
   if the series is convergent and $f(s) = 0$ otherwise. Prove that f is integrable and that
   $$\int f\, d\mu = \sum_{n=1}^\infty \int f_n\, d\mu.$$

6. Let $f \in L^1(0,\infty)$ and define
$$h(x) = \int_0^\infty (x+y)^{-1} f(y) dy$$
for $x > 0$.

(a) Show that h is differentiable at all $x > 0$.
(b) Show that $h' \in L^1(r, \infty)$ for any fixed $r > 0$.
(c) Must the statement of (b) always be true for $r = 0$?

7. Let μ be the signed measure with distribution function

$$F(x) = \begin{cases} 2 - x^2 \text{ if } -1 < x \leq 1, \\ 0 \text{ else.} \end{cases}$$

Find the distribution function of the total variation measure $|\mu|$.

8. Let ν be a finite signed measure and μ a positive measure. Show that $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$. Give an example showing that this may fail if ν is not finite.

Part 2: Complex and Functional Analysis

9. Calculate

$$\int_{|z|=1} (e^{2\pi z} + 1)^{-1} dz$$

where the integration path is traced counterclockwise.

10. Let f be an analytic function on an open, connected subset G of $\mathbb{C}$. Prove that the following statements are equivalent:

(a) f is identically zero on G,
(b) there exists a point $a \in G$ with $f^{(n)}(a) = 0$ for all $n \geq 0$,
(c) $\{z \in G : f(z) = 0\}$ has a limit point in G.

(You may use without proof facts about the expansion of f into power series in any disk contained in G. You may not use the uniqueness theorem.)

11. Let D be the open unit disk and $A = \{z : 1 < |z| < 2\}$.

(a) Let $g : D \backslash \{0\} \to A$ be analytic. What type of singularity of g is the point $z = 0$?
(b) Prove that there is no one-to-one conformal map of $D \backslash \{0\}$ onto A.

12. Let A be a Lebesgue measurable set in $\mathbb{R}$ with $0 < m(A) < \infty$. Show that for $1 < p < q \leq \infty$ we have $L^q(A) \subseteq L^p(A)$.

13. Let H be a Hilbert space and A a non-empty closed convex subset of H. Show that A has a unique element of minimal norm.

14. Let X, Y be normed spaces with $X \neq 0$ and denote by $B(X, Y)$ the vector space of bounded linear transformations from X to Y. Assume that $B(X, Y)$ is a Banach space with the usual norm $\|T\| = \sup\{|T(x)| : x \in X, \|x\| = 1\}$. Show that then Y is a Banach space.