Analysis Preliminary Exam
August 2020

Measure Theory and Integration

1. A set $E \subseteq [0,1]$ has the property that there exists $0 < d < 1$ such that for every $(\alpha, \beta) \subset [0,1]$,
$$m(E \cap (\alpha, \beta)) > d(\beta - \alpha).$$
Prove that $m(E) = 1$. (m is Lebesgue's measure)

2. Let Q be the set of rational numbers in $(0,1]$. Let M be the algebra consisting of finite unions of sets of the form $Q \cap (a, b]$, where $0 \leq a < b \leq 1$. Define a finitely-additive set function μ on M by
$$\mu(Q \cap (a,b]) = b - a, \quad \text{ and } \mu\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i),$$
where the A_i are pairwise disjoint and for $1 \leq i \leq n$, $A_i = Q \cap (a_i, b_i]$ for some $a_i, b_i \in [0,1]$. Is μ countably additive on M? Justify your answer.

3. Give an example of a sequence $\{f_n\}_{n=1}^\infty$ of non-negative functions on the interval $[0,1]$ that satisfies the following properties:
   (i) f_n is continuous for $n = 1, 2, 3, \ldots$.
   (ii) For each $x \in [0,1]$, $\{f_n(x)\}_{n=1}^\infty$ is unbounded.
   (iii) $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$.

4. Let $f : [a,b] \to [0,\infty)$ be Lebesgue measurable. Define the function ω on $[0,\infty)$ by
$$\omega(y) = m\{x : f(x) > y\}.$$
Prove that ω is right continuous (and hence measurable), and that
$$\int_a^b f(x)dx = \int_0^\infty \omega(y)dy.$$

5. Consider the function $f(x) = \frac{1}{\sqrt{x}}$ on $[0,1]$.
   (i) Show that f is is measurable on $[0,1]$.
   (ii) Calculate $\int_{[0,1]} \frac{1}{\sqrt{x}} dm$. [Notice: f is not Riemann-integrable on $[0,1]$.]

6. Consider functions $f, g : [-1, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (i) Find $\overline{D}f(0)$, $\underline{D}f(0)$, $\overline{D}g(0)$ and $\underline{D}g(0)$.

   (ii) Determine if f and g are of bounded variation on $[-1, 1]$.

7. Consider the measure space $([0, 1], \mathcal{F}|_{[0,1]}, m)$, where m is the Lebesgue measure and $\mathcal{F}$ is the Lebesgue measurable sets, and let ν be the counting measure on $\mathcal{F}|_{[0,1]}$. Show that

   (i) $m << \nu$, and

   (ii) there is no function $f : [0, 1] \to \mathbb{R}$ for which $m(E) = \int_E f d\nu$ for all $E \in \mathcal{F}|_{[0,1]}$.

8. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ be $\mathcal{A}$- and $\mathcal{G}$-measurable functions, respectively. Prove that:

   (i) the function $h(x, y) = f(x)g(y)$ is $\mathcal{A} \times \mathcal{G}$-measurable

   (ii) if f and g are integrable, so is h and

$$\int_{X \times Y} h \, d(\mu \times \nu) = (\int_X f \, d\mu) \, (\int_Y h \, d\nu).$$

Complex, Functional and Harmonic Analysis

1. Let $\{f_n\}$, f be Lebesgue measurable functions on $\mathbb{R}$ such that $f_n \to f$ almost everywhere. If there exists a constant $C < \infty$ and $p > 1$ such that $\|f_n\|_p \leq C$ for every n, show that $f_n \to f$ in L^q for every $1 \leq q < p$.

2. Let f be a non-negative function such that $f \in L^p(0, 1)$ for every $p \geq 1$. If $\|f\|_p^p = \|f\|_1$ for every $p > 1$, prove that f is the characteristic function of a measurable set $E \subseteq (0, 1)$.

3. Let p, q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(\mathbb{R})$. For $f \in L^p$ and $y \in \mathbb{R}$, define the function $T_y f$ by $T_y f(x) = f(x - y)$. Let

$$Lf(y) = \int (T_y f)(x) g(x) dx.$$

   Show that L is a continuous linear operator from L^p to L^∞.

4. Use Morera's theorem to show that f defined by

$$f(z) = \int_0^\infty e^{-zt} t^{-3} \sin^3(t) dt$$

is analytic in $\Re z > 0$.

5. Let $G \subseteq \mathbb{C}$ be a region. If $f : G \to \mathbb{C}$ is analytic except for poles, show that the poles of f cannot have a limit point in G.

6. Let $f : \mathbb{D} \to \mathbb{D}$, where $D = \{z : |z| < 1\}$. Assume that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = cz$ for some constant c with $|c| = 1$. (Hint: Consider $g(z) = z^{-1} f(z)$.)

7. Let $T \in B(l_2(\mathbb{C}))$ be defined by $T(x) = (\alpha_i x_i)$, where $(\alpha_i) \in l_\infty(\mathbb{C})$ is a fixed sequence. Prove that

   (i) T is linear and continuous with $\|T\| = \|\alpha\|_\infty$.

   (ii) If $\alpha = (\alpha_i),\ \alpha_i \in \mathbb{R}$, for all $i \geq 1$, then T is Hermitian.

8. Consider $C[0,1], \| \|_\infty)$ and let $T \in B(C[0,1])$ be defined by $Tf(x) = \int_{[0,x]} f(t)dt$. Prove that T is a compact operator.

9. Let (X, d) be a metric space and M be a subset of X. Prove that

   (i) If $A \subset M$ is nowhere dense in M, then A is nowhere dense in X.

   (ii) If $A \subset M$ is first category in M, then A is first category in X.

   (iii) If $A \subset M$ is second category in M, does it imply that A is second category in X? Justify your answer.