Analysis Preliminary Exam, January 2020

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Measure Theory and Integration

1. Show that if $E_1 \cup E_2$ is Lebesgue measurable and $m(E_2) = 0$, then E_1 is Lebesgue measurable.

2. For every $\epsilon > 0$ and every subset E of the real numbers, define
$$\mu_\epsilon(E) = \inf \sum_{n=1}^{\infty} (m(I_n))^{1/2}$$
where the infimum is taken over all countable collections of open intervals $\{I_1, I_2, \ldots\}$ with length less than ϵ that cover E.
   (a) Show that μ_ϵ is an outer measure on $\mathbb{R}$.
   (b) Let $\mu(E) := \sup_{\epsilon>0} \mu_\epsilon(E) = \lim_{\epsilon\to 0+} \mu_\epsilon(E)$. Show that $\mu(0,1) = +\infty$. Hint: $m(I_n)^{1/2} = \frac{m(I_n)}{m(I_n)^{1/2}} \geq \epsilon^{-1/2} m(I_n)$.

3. Let $A \subset \mathbb{R}$ such that $m(A) = 0$. Let $B = \{x \in \mathbb{R} : x = y^2 \text{ for some } y \in A\}$. Prove that $m(B) = 0$.

4. Let $f \in L^1[0,1]$. Show that
$$\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx$$
exists and find its value.

5. Give an example of a Borel measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y)$ is Lebesgue integrable with respect to y for every fixed x, and integrable with respect to x for every fixed y, the functions $x \to \int_{\mathbb{R}} f(x,y)dy$ and $y \to \int_{\mathbb{R}} f(x,y)dx$ are Lebesgue integrable, but
$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)dy\right] dx \neq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)dx\right] dy.$$

6. Let $f : [a,b] \to [0,\infty)$ be Lebesgue measurable. Define the function ω on $[0,\infty)$ by
$$\omega(y) = m\{x : f(x) > y\}.$$
Prove that ω is right continuous (and hence measurable), and that
$$\int_a^b f(x)dx = \int_0^\infty \omega(y)dy.$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function and $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Prove that $g \circ f$ is absolutely continuous.

8. Let E be a Lebesgue measurable set and let m be Lebesgue measure. The density δ of E at a point $x \in \mathbb{R}$ is defined by

$$\delta(x) = \lim_{h \to 0} \frac{m[(x-h, x+h) \cap E]}{2h},$$

provided that this limit exists.

   (a) Prove that $\delta(x) = \chi_E(x)$ for almost every x. Hint: Consider the integral of $\chi_E(x)$.

   (b) Let $E = \cup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n} \right)$. Calculate the density of E at 0.

Complex, Functional and Harmonic Analysis

1. (a) Show that there are no non-real solutions of $\cos z = 0$.

   (b) Show that there exist constants a_k so that for all $|z| < \pi/2$

$$\frac{1}{\cos z} = 1 + \sum_{k=1}^{\infty} \frac{a_k}{(2k)!} z^{2k}.$$

2. Calculate

$$\int_{|z|=1} (e^{2\pi z} + 1)^{-1} dz$$

where the integration path is traced counterclockwise.

3. (a) State Schwarz's Lemma.

   (b) Let $f : D \to D$, where $D = \{z : |z| < 1\}$. Assume $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = cz$ for some constant c with $|c| = 1$. (Hint: Consider $z^{-1}f(z)$.)

4. Prove that there is no one-to-one conformal map of $D \backslash \{0\}$ onto A.

5. Let p, q, r be positive real numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let $f \in L^p$, $g \in L^q$ and $h \in L^r$. Prove the generalized Hölder's inequality,

$$\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r.$$

6. Show that the triangle inequality fails for weak L^p spaces: Find two functions f, g in $L^p[0,1]$ (with Lebesgue measure) such that $[f]_p = 1$, $[g]_p = 1$ and $[f+g]_p > 2$.

7. Let $f \in L^1(\mathbb{T})$ such that $\{\widehat{f}(n)\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$. Show that there is a continuous function g such that $g = f$ a.e.

8. State and prove Riemann-Lebesgue Lemma.

9. Consider the space c_0 of all real sequences converging to zero.

   (a) State the definition of a separable normed space.

   (b) Prove or disprove: the space c_0 is separable.

10. Let H be a Hilbert space and A a non-empty closed convex subset of H. Show that A has a unique element of minimal norm.

11. Prove that a necessary and sufficient condition for a normed space X to be complete is that for every sequence of vectors $\{x_n\} \subset X$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the series $\sum_{n=1}^{\infty} x_n$ converges to an element $x \in X$.

12. Consider the space $(L_2[0,1], \| \|_2)$ and let $T : L_2[0,1] \to L_2[0,1]$ be a map defined as, for all $f \in L_2[0,1]$,
$$Tf(x) = \int_{[0,1]} K(x,y) f(y) dy,$$
where $K : [0,1]^2 \to \mathbb{R}$ is a continuous function. Prove that

    (a) T is a bounded linear map.

    (b) What is $\|T\|$? Justify your answer.