Analysis Preliminary Examination
August 2024

Submit six of the following problems. Start every problem on a new page, number your pages and write your student ID on each page. **Do not** write your name.

Lebesgue measure is denoted m *and unless stated otherwise* $(X, \mathcal{M}, \mu)$ *is a generic measure space.*

1. Let $E \in \mathcal{F}$ be a bounded set with $m(E) \geq 0$. Prove that if $\Lambda \subset \mathbb{R}$ is a countably infinite set such that $\bigcup_{\lambda \in \Lambda} \lambda + E$ is a bounded disjoint union, then $m(E) = 0$.

2. Let $\{f_n\}$ be a sequence of measurable functions on X. Prove that the set
$$\{x \in X : \lim_n f_n(x) \text{ exists}\} \quad \text{is measurable.}$$

3. a) Give the definition of a measurable function.
b) Show that if f is measurable and $g = f$ a.e., then g is also measurable.
c) True or false: if f^2 is measurable, then so is f. Justify your answer.

4. a) State monotone convergence theorem.
b) Is monotonicity necessary? Justify your answer (with an example).

5. For $a > 0$ we define the function
$$f(a) = \int_0^\infty e^{-at} \frac{\sin t}{t}\, dt.$$
Justify the existence of the limit $\lim_{a \to \infty} f(a)$ and find its value. (Say which convergence theorem you are using).

6. Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu$ be counting measure. Consider the function $f : X \times Y \to \mathbb{R}$, where
$$f(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$
Prove that $\int |f| d(\mu \times \nu) = \infty$, and both the iterated integrals $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ exist and are unequal.

7. Prove that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then its of bounded variation (hence differentiable almost everywhere).

8. On $\mathbb{R}$ we consider the Lebesgue-Stieltjes measure μ_F with distribution function
$$F(x) = \begin{cases} x + [x] & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$
where $[x]$ is the integer part of x.

(a) Show that F is right-continuous.

(b) Calculate $\mu_F[4, 8]$ and $\mu_F[3, 7)$.

(c) Find a set $A \subset \mathbb{R}$ with Lebesgue measure 0 such that $\mu_F(A) > 0$.