Analysis Preliminary Examination
January 2025

Submit six of the following problems. Start every problem on a new page, number your pages and write your student ID on each page. **Do not** write your name.

Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Consider the measure space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \nu)$ where ν is the counting measure. Define for $A \subset \mathbb{Z}$,
$$\mu^*(A) = \nu(A)^{1/3}.$$
Prove that μ^* is an outer measure, and that the only measurable sets are $\emptyset$ and $\mathbb{Z}$.

2. Let $\{F_\alpha\}$ be an infinite (countable or uncountable) family of closed subsets of $[0,1]$. Assume that the Lebesgue measure of every finite intersection $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$ is at least $1/2$. Prove that the Lebesgue measure of the full intersection $\cap_\alpha F_\alpha$ is also at least $1/2$.

3. Prove that a finitely additive set function is a measure iff it is continuous from below.

4. a) Let $\{f_n\}_n \subset L^+[0,1]$ be a sequence of functions such that $f_n \downarrow f$ pointwise. Prove that, if $\int f_1 dm < \infty$, then $\int f dm = \lim_n \int f_n dm$.
b) Is the assertion valid when the assumption $\int f_1 dm < \infty$ is removed? Justify your answer.

5. a) State Chebychev's Theorem.
b) Show that if $f \in L_1(\mathbb{R})$ and $E_n = \{x \in \mathbb{R} : \ |f(x)| > n\}$, then
$$\lim_n \int_{E_n} f(x) dm = 0.$$

6. Let $f, g \in L^1(m)$, where m is Lebesgue measure on $[0,1]$. Let
$$G(x) = \int_0^{\sqrt{1-x^2}} g(y) dm(y) \quad F(y) = \int_0^{\sqrt{1-y^2}} f(x) dm(x).$$
Use Fubini's theorem to prove that
$$\int_0^1 f(x) G(x) dm(x) = \int_0^1 F(y) g(y) dm(y).$$

7. Let $f_n, g_n : [0,1] \to [0, \infty)$ be measurable functions. Assume that f_n converges in measure to zero on the set $[0,1]$, and that for every $n \in \mathbb{N}$,
$$\int_{[0,1]} g_n < 1.$$
Prove that $f_n g_n$ converges in measure to 0 on the set $[0,1]$.

8. a) Justify the identity

$$\int_0^1 [\sum_{n=0}^{\infty}(-x)^n] dm = \sum_{n=0}^{\infty} \int_0^1 (-x)^n \ dm.$$

b) Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$.

9. a) Give the definition of a function of bounded variation.
b) Provide an example of a continuous function on [0,1] which is not a function of bounded variation.