ANALYSIS EXAM August 2004

1. a) Give the definition of a complete metric space.
   b) Let (X, ρ) be a metric space (which may or may not be complete). Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in X. Define a numerical sequence $\{a_n\}$ by $a_n = \rho(x_n, y_n)$. Show that the sequence $\{a_n\}$ is convergent.

2. a) Give the definition of a compact metric space.
   b) Let (X, ρ) be a compact metric space. Suppose a function $f : X \to X$ satisfies the following property: $\rho(f(x), f(y)) < \rho(x, y)$ for all pairs of points $x, y \in X$, $x \neq y$. Show that there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.
   Hint: You may want to consider the auxiliary real-valued function
   $g(x) = \rho(x, f(x))$.

3. It is known (and was proved in the analysis course) that an increasing real-valued function on an interval $[a, b]$ on the real line is differentiable almost everywhere. Prove or disprove the analog of this statement for the entire real line. In other words, let f be a real-valued function on the real line such that $f(x) \leq f(y)$ if $x < y$. Is it true that f must be differentiable almost everywhere on the real line?

4. Let $1 \leq p < \infty$. It is well known that $L^p[0,\ 1]$ is a vector space. In particular, the sum of two L^p-functions must also be in L^p.
   Prove this statement, i.e., prove that if $f \in L^p[0,\ 1]$ and $g \in L^p[0,\ 1]$, then $(f + g) \in L^p[0,\ 1]$.
   Remark: do not use Minkowski inequality; the traditional proofs of Minkowski inequality use the above statement.

5. Let $\{f_n\}$ be a sequence of functions in $L^2[0,\ 1]$, and let f be also in $L^2[0,\ 1]$. Suppose that $\{f_n\}$ converges to f in the metric of the space $L^2[0,\ 1]$. Does this imply that f_n converges to f almost everywhere? Prove or give a counterexample.

6. a) Define the indefinite integral of an integrable function on $[0, 1]$.
   b) Define functions of bounded variation.
   c) If f is a function of bounded variation on $[0, 1]$, is it true that it is the indefinite integral of its a.e. derivative? Why? If your answer is negative, for what type of functions the answer is affirmative?

7. State the Bounded Convergence Theorem. Show, by an example, that the boundedness requirement in this theorem is essential.

8. a) State the definitions of sets of first and second category in a metric space (M, d).
   b) Show that the plane $\mathbb{R}^2$ is not a countable union of lines (a line is a set $\{(x, y) : ax + by = c,\ a, b, c \in \mathbb{R}\}$, where a and b are not both zero).

9. For a sequence $\{A_n\}$ of measurable subsets of $\mathbb{R}$, define

$$\liminf A_n = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i, \quad \text{and} \quad \limsup A_n = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i.$$

Show, for such a sequence $\{A_n\}$, that

a) $\mu(\liminf A_n) \le \mu(\limsup A_n)$,

b) $\mu(\limsup A_n) \ge \limsup \mu(A_n)$ if $\mu(\cup_{n=1}^{\infty} A_n) < \infty$.

(μ is the Lebesgue measure on $\mathbb{R}$.)

10. Show that a measurable function f is integrable if and only if $|f|$ is integrable. Is there a nonintegrable function f for which $|f|$ is integrable?