Analysis Qualifying Exam
May 2019

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

1 Real Analysis

Throughout, m denotes Lebesgue measure. You may use without proof the Lebesgue dominated convergence theorem and Fubini's theorem.

1. (a) Give the definition of a Lebesgue measurable set in $\mathbb{R}^n$.

   (b) Show that a set $A \subseteq \mathbb{R}^n$ of outer measure zero is measurable.

2. Let μ_F be the (signed) Borel measure with distribution function

$$F(x) = \begin{cases} x & \text{if } x < 0, \\ x - 1 & \text{if } 0 \le x < 1, \\ x - 2 & \text{if } x \ge 1. \end{cases}$$

   (a) Find a Hahn-decomposition of $\mathbb{R}$ for μ_F.

   (b) Find the Radon-Nikodym decomposition of μ_F with respect to Lebesgue measure.

3. Prove or give a counterexample: If (f_n) converges to f in $L^1[0,1]$, then $f_n \to f$ pointwise.

4. (a) Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be Lebesgue measurable.

   (b) Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable for $n \in \mathbb{N}$. Show that $g = \inf_n f_n$ is measurable.

5. Let $\mathcal{A}$ be an algebra of sets that is closed under countable increasing unions. Show that $\mathcal{A}$ is a σ-algebra.

6. (a) Give the definition of a function of bounded variation on the interval $[0,1]$

   (b) If f, g are functions of bounded variation on $[0,1]$, is fg a function of bounded variation on $[0,1]$? Justify your answer.

7. (a) State the Dominated Convergence Theorem.

   (b) Let $f_n(x) = \dfrac{nx \ln x}{1 + n^2x^2}$. Find $\lim_{n\to\infty} \int_{[0,1]} f_n dm$. Justify your answer.

8. Using Fubini's Theorem and the fact that $1/x = \int_0^\infty e^{-xt} dt$ for $x > 0$, compute the integral $\int_0^\infty \frac{\sin x}{x} dx$. Justify your work.

2 Complex, Functional, and Harmonic Analysis

9. Use Cauchy's integral formula to prove Liouville's theorem that every bounded function is constant.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on the upper half plane and continuous on the closed half plane $\Im z \geq 0$ and is real valued on $\mathbb{R}$. Define

$$g(z) = \begin{cases} f(z) & \text{if } \Im z \geq 0, \\ \overline{f(\bar{z})} & \text{if } \Im z < 0. \end{cases}$$

    Show that g is entire.

11. (a) Let G be the open unit disk. If $u : G \to \mathbb{R}$ is harmonic, show that u has a harmonic conjugate.

    (b) Find an example illustrating that the statement of (a) is false if G is the punctured unit disk.

12. Let (X, μ) be a measure space with $\mu(X) = 1$. Let $f, g \in L^1(X, \mu)$ be positive functions satisfying $f(x)g(x) \geq 1$ μ-a.e. Use Hölder's inequality to show that

$$\left(\int_X f d\mu \right) \left(\int_X g d\mu \right) \geq 1.$$

13. Prove that if $f \in weakL^p$ and $\mu\{x : f(x) \neq 0\} < \infty$, then $f \in L^q$ for all $q < p$.

14. Let $f \in L^1(\mathbb{T})$ and let $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ its Fourier coefficients. State and prove the Riemann-Lebesgue Lemma.

15. Let $\phi \in L^1(\mathbb{R}^n)$ such that $\int \phi = 1$. Define $\phi_t(x) = t^{-n}\phi(x/t)$. Show that for every $f \in L^p$ we have

$$\lim_{t\to 0} \|\phi_t \star f - f\|_p = 0.$$

16. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear map. Prove that $\|T\|^2 = \|TT^*\| = \|T^*\|^2$.

17. Let $\mathcal{X}$ be a Banach space and $T : \mathcal{X} \to \mathcal{X}$.

    (a) State the Open Mapping Theorem.

    (b) If T is an invertible bounded linear map, prove that so is T^{-1}.

18. (a) State the definition of a reflexive normed space.

    (b) Let $\mathcal{X}$ be a reflexive Banach space and $\mathcal{Y}$ be a closed subspace. Prove that $\mathcal{Y}$ is also a reflexive Banach space.