Problems for Preliminary Exam
Applied Mathematics
May 2018

Part I
All problems have 10 points.

1. Consider

$$\dot{x} = f(x),$$

with $f \in C^1(\mathbb{R})$. Show that a fixed point x_0 is exponentially stable if $f'(x_0) < 0$ and unstable if $f'(x_0) > 0$.

2. Let $x^{(k)} = f(x, x^{(1)}, x^{(2)}, \ldots, x^{(k-1)})$ be an autonomous equation (or system). Show that if $\phi(t)$ is a solution then so is $\phi(t - t_0)$, where $t_0 \in \mathbb{R}$.

3. Show that $\Phi(t, x) = e^t(1 + x) - 1$, is a flow. Find an autonomous system corresponding to this flow.

4. Consider the linear autonomous systems

$$\text{(a)} \quad \dot{x} = -x, \quad \dot{y} = -2y, \quad \text{and} \quad \text{(b)} \quad \dot{x} = 4y, \quad \dot{y} = -x.$$

Without using any Lyapunov function, show that the equilibrium point $(0, 0)$ of system (a) is asymptotically stable, but that the critical point $(0, 0)$ of system (b) is stable but not asymptotically stable.

5. A function $A : \mathbb{R} \to \mathrm{GL}(n; \mathbb{C})$ is called a **one-parameter subgroup** of $\mathrm{GL}(n; \mathbb{C})$ if

- A is continuous,
- $A(0) = I$,
- $A(t + s) = A(t)A(s)$, for all $t, s \in \mathbb{R}$.

If A is a one-parameter subgroup of $\mathrm{GL}(n; \mathbb{C})$; show that there exists a unique $n \times n$ complex matrix X such that

$$A(t) = e^{tX}.$$

[Notation: $\mathrm{GL}(n; \mathbb{C})$, the general linear group of degree n, is the set of $n \times n$ complex invertible matrices, together with the operation of ordinary matrix multiplication.]

Part II
All problems have 10 points.

1. Compute the solution to

$$uu_x + yu_y = x, \quad u(x,1) = 2x.$$

Clearly state for which (x, y) the solution is defined.

2. Consider the following PDE

$$\Delta u - q(x)u = 0, \quad q(x) \geq 0, \quad x \in \Omega,$$

where Ω is a bounded connected region with a sufficiently nice boundary. Establish the uniqueness theorem for this problem for the Dirichlet boundary conditions

$$u(x) = g(x), \quad x \in \partial\Omega.$$

3. Let L be a linear differential operator with $\mathcal{C}^\infty$ coefficients. Write down a definition of a *weak solution* to the differential equation

$$Lu = f.$$

Using your definition show that $u(t,x) = H(x-t)+H(x+t)$ (here H is the Heaviside function) is a weak solution to the one-dimensional wave equation $u_{tt} = u_{xx}$ in all $\mathbb{R}^2$.
(Hint: using new coordinates $\xi = x + t, \eta = x - t$ or similar may help.)

4. Which problem is called *well-posed*?
Show that the initial-boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0, \quad y > 0, \quad 0 < x < \pi,$$

with the following boundary conditions

$$u(0,y) = u(\pi,y) = 0, \quad y > 0,$$

and initial conditions

$$u(x,0) = f(x), \quad u_y(x,0) = g(x), \quad x \in [0,\pi],$$

is *not* well-posed (i.e., it is *ill-posed*).
(Hint: take $f(x) = 0$ and $g(x) = e^{-\sqrt{n}} \sin nx$, where n is an odd integer.)

5. Solve the following initial-boundary value problem for the heat equation

$$u_t = \alpha^2 u_{xx}, \quad x > 0, \quad t > 0,$$

with the initial condition $u(x,0) = g(x)$, $x > 0$ and the boundary condition $u(0,t) = 0$ for $t > 0$. Here g is a continuous and bounded function with $g(0) = 0$.