Analysis Preliminary Exam
Measure Theory and Integration

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page. **Do not write your name.**

1. Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, let $\mathcal{M}_\alpha$ be a σ-algebra on X_α, and let $X = \prod_{\alpha \in A} X_\alpha$.
   (a) Write the definition of the product σ-algebra $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ on X.
   (b) Prove that, if A is countable, the product σ-algebra is generated by
   $$\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}.$$
   (c) Why do the sets in (b) may fail to generate the product σ-algebra if A is uncountable? Explain.

2. (a) State the definition of Lebesgue outer measure m^*.
   (b) Prove that $\forall\ A \subset \mathbb{R}$ Lebesgue measurable and $\forall\ \alpha \in \mathbb{R}$, $m^*(A) = m^*(\alpha + A)$.

3. Let $E \subset \mathbb{R}$. Prove that the following are equivalent:
   (a) E is Lebesgue measurable.
   (b) For every $\varepsilon > 0$ there is an open set U such that $m^*(E \Delta U) < \varepsilon$.
   (c) For every $\varepsilon > 0$ there is a closed set F such that $m^*(E \Delta F) < \varepsilon$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions.
   (a) Write the definition of convergence in measure of f_n to f.
   (b) Assume that f_n converges in measure to a function f. Prove that there exists a subsequence $\{f_{n_j}\}$ that converges to f for almost every $x \in X$.

5. (a) State Fatou's Lemma.
   (b) State the Monotone Convergence Theorem.
   (c) Show by an example that the monotonicity assumption is essential.
   (d) Prove the Monotone Convergence Theorem. (Hint: Use Fatou's Lemma.)

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of non-negative functions in $L^1(X)$. Assume that f_n converges pointwise to a function $f \in L^1(X)$. Show that
   $$\int_X f_n d\mu - \int_X f d\mu - \|f - f_n\|_{L^1} \to 0,$$
   as $n \to \infty$. Hint: Apply the Dominated Convergence Theorem to $\min(f, f_n)$.

The last two questions are in the next page.

7. (a) Show that every function $f : \mathbb{R} \to \mathbb{R}$ of bounded variation is bounded, and that the limits $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ exist.

   (b) Give an example of a bounded, continuous, compactly supported function that is not of bounded variation.

8. (a) State the definition of Lebesgue decomposition of a measure.

   (b) Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0, 1]$. Prove that the counting measure on $\mathcal{B}$ has no Lebesgue decomposition w.r.t. Lebesgue measure m on $\mathcal{B}$.