Analysis Preliminary Examination
May 2016

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis

Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^+$ and $\int f\, d\mu < \infty$ then for every $\varepsilon > 0$ there is $E \in \mathcal{M}$ with $\mu(E) < \infty$ such that $\int_E f\, d\mu > \left(\int f\, d\mu\right) - \varepsilon$.

2. Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$.
   (a) Prove that μ_E is a measure on $(X, \mathcal{M})$.
   (b) Is it true that μ_E is complete if μ is (Justify your answer)?

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Prove that if μ and ν are σ-finite measures then $\mu \times \nu$ is a σ-finite measure.

4. Let $\{f_n\}$ and $\{g_n\}$ be sequences of integrable functions, and let f, g be integrable functions such that $f_n \to f$ *a.e.* and $g_n \to g$ *a.e.*. Assume that $|f_n(x)| \leq g_n(x)$ for all x, and that $\int g_n\, dm \to \int g\, dm$. Prove that
$$\lim_{n\to\infty} \int f_n\, dm = \int f\, dm.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let L^+ be the set of all positive measurable functions on X. For $f \in L^+$, we define
$$\nu(E) = \int_E f\, d\mu.$$
Show that ν defines a measure on X, and that if $g \in L^+$,
$$\int g\, d\nu = \int fg\, d\mu.$$

6. Let μ be the counting measure on $\mathbb{N}$. We define a function f on $(\mathbb{N} \times \mathbb{N}, \mu \times \mu)$ by $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$ and $f(m,n) = 0$ otherwise. Show that the iterated integrals $\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(m,n)\, d\mu(m)\right)\, d\mu(n)$ and $\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(m,n)\, d\mu(n)\right)\, d\mu(m)$ exist but have different values, and explain why this does not contradict Fubini's theorem.

7. Let $F : \mathbb{R} \to \mathbb{R}$. There is a constant M such that $|F(x) - F(y)| \leq M|x - y|$ for all x, y if and only if F is absolutely continuous and $|F'| \leq M$ a.e.

8. Let dF be the Lebesgue-Stieltjes measure with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ (e^x - 1)/2 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x \end{cases}$$

   (a) Prove that F is not absolutely continuous with respect to Lebesgue measure.

   (b) Find the Radon-Nikodym decomposition of dF with respect to Lebesgue measure, i.e., find $f \in L^1(\mathbb{R}, m)$ and a measure μ with $\mu \perp m$ so that $dF = f dm + d\mu$.

Part 2: Complex and Functional Analysis

1. Prove the Riesz Representation Theorem for Hilbert spaces: Every Hilbert space H is isometrically isomorphic to its dual H^*.

2. Let X, Y be Banach spaces, and let $T : X \to Y$ be a linear operator such that for every $f \in Y^*$, $f \circ T \in X^*$. Use the Closed Graph Theorem to prove that T is continuous.

3. Let X be a normed vector space and denote by X^* the dual space. Prove that X^* is complete.

4. Assume that $f = u + iv$ is a complex function which is differentiable at x_0. By taking limits parallel to the coordinate axes prove that the Cauchy-Riemann equations are satisfied for f at x_0. (The C-R equations are the system of equations $u_x = v_y$ and $u_y = -v_x$).

5. Let γ be the positively oriented unit circle and compute

$$\frac{1}{2\pi i} \int_\gamma \frac{e^z - e^{-z}}{z^4}\, dx.$$

   (You may need to use the power series for $e^z = \sum \frac{z^n}{n!}$).

6. Recall that if $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ on $D(A; R)$ and $0 < r < R$ then

$$\sum_{n=1}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|2\, d\theta.$$

   Use this fact to prove that if f is holomorphic in a region containing $D(a; R)$ then $|f(a)| \leq \max\{|f(a + e^{i\theta})| : \theta \in [0, 2\pi]\}$.