ANALYSIS EXAM

QUESTIONS.

1. a) Show that every closed set on the real line can be represented as an intersection of a countable collection of open sets (in other words, show that every closed set is a G_δ-set).

b) Give an example of a G_δ-set which is neither open, nor closed.

2. a) Give a definition of a measurable function on the real line.

b) Let f be a real-valued function on the real line. Which of the following statements are true? Justify your answers.

(i) If f is measurable, then f^2 is measurable.

(ii) If f^2 is measurable, then f is measurable.

3. a) Let E denote the set of all irrational numbers on the interval [0, 1]. What is the Lebesgue measure of E? Justify your answer.

b) Let E be the same as in part a) of this problem. Does there exist a **closed** set F of positive Lebesgue measure such that $F \subseteq E$? In other words, is there a **closed** set F on [0, 1] of positive Lebesgue measure, which consists only of irrational numbers?

4. a) State Dominated Convergence Theorem.

b) Let $\{f_n\}$ be a sequence of real-valued integrable functions on [0,1] such that $0 \leq f_{n+1} \leq f_n$ a.e. for all $n \geq 1$. Show that $f_n \downarrow 0$ a.e. holds if and only if $\int_{[0,1]} f_n dm \downarrow 0$ (here m denotes the Lebesgue measure).

5. Let $f : [0, \infty) \to \mathbb{R}$ be a function Riemann integrable on every closed subinterval of $[0, \infty)$. Let $\int_{[0,\infty)} f(x)dx$ denote the Lebesgue integral of f on $[0, \infty)$ and $\int_0^\infty f(x)dx$ denote the improper Riemann integral of f. Prove or disprove:

a) If $\int_0^\infty f(x)dx$ exists, then so does $\int_{[0,\infty)} |f(x)|dx$.

b) If $\int_0^\infty |f(x)|dx$ exists, then $f \in L_1[0, \infty)$.

6. a) Prove that $L_p[0,1] \subset L_1[0,1]$ for $1 \leq p \leq \infty$.

b) Is the inclusion $L_2[0,1] \subset L_1[0,1]$ proper? Justify your answer.

7. a) Give the definition of a function of bounded variation on the interval $[0,1]$.

b) Show that if f and g are functions of bounded variation on $[0,1]$, then so is fg.

8. a) Give the definition of an absolutely continuous function on the interval $[0,1]$.

b) Let $f(x) = \sqrt{x}$. Is f an absolutely continuous function on the interval $[0,\ 1]$?

Justify your answer.

9. a) State Stone-Weierstrass Theorem.

b) If $f : [0,1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 x^n f(x)dx = 0$ for $n = 0,\ 1,\ 2,\ \ldots,$ then show that $f(x) = 0$ for all $x \in [0,1]$.

10. a) Give a definition of a bounded linear functional on a Banach space X.

b) Describe the general form of a bounded linear functional on the space $L_p[0,1], \quad 1 \leq p < \infty$.